

Practical Probability: Casino Odds and Sucker Bets

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Abstract

Gambling casinos are there to make money, so in almost every instance, the games you can bet on will, in the long run, make money for the casino. To make people gamble there, however, it is to the casino's advantage to make the bets appear to be "fair bets", or even advantageous to the gambler. Similarly, "sucker bets" are propositions that look advantageous to one person but are really biased in favor of the other. In this article, we'll examine what is meant by a fair or biased bet, and we will look in detail at some casino games and sucker bets.

1 Introduction

One of these days in your travels, a guy is going to come up to you and show you a nice brand-new deck of cards on which the seal is not yet broken, and this guy is going to offer to bet you that he can make the Jack of Spades jump out of the deck and squirt cider in your ear. But, son, do not bet this man, for as sure as you are standing there, you are going to end up with an earful of cider.

—Damon Runyon

There are plenty of sucker bets that simply depend on one person knowing something that the other person doesn't. For example, if someone offers to play "dollar bill poker" with you, where each of you pulls a dollar bill out of your wallet and the one whose serial number can be interpreted as the best poker hand wins, it may be that that person has been saving bills that have great poker hands and his wallet is stuffed with them. Instead of "loaded" dice he's effectively using "loaded" dollar bills. What we're interested in here, however are bets where the gambling instruments (dice, cards, flipped coins, et cetera) are fair, but the bet is structured in a way that the most likely event is surprising.

As a first example, consider the following. Six cards are selected from a deck: two kings and four aces. The bet is the following: the deck of six is shuffled after which the top two cards are selected. If both cards are aces, you win; if at least one is a king, I win. Two-thirds of the cards are aces, so it may seem that you will usually win, but that is not the case. Let's see why.

There are six cards and we are choosing two of them, so there are $\binom{6}{2} = 15$ ways to do this. If the cards are A_1, A_2, A_3, A_4, K_1 and K_2 , here is a complete list of the possible pairs:

A_1A_2	A_1A_3	A_1A_4	A_1K_1	A_1K_2
A_2A_3	A_2A_4	A_2K_1	A_2K_2	A_3A_4
A_3K_1	A_3K_2	A_4K_1	A_4K_2	K_1K_2

If we count the number of the 15 pairs above that contain at least one king, we see that there are 9 of them, and only 6 that contain only aces. Thus $3/5$ of the time there will be at least one king.

Another way to see this is as follows: $4/6$ of the time, the first card will be an ace. If an ace is selected as the first card, there remain 3 aces and 2 kings, so the second card will be an ace $3/5$ of the time. The probability that *both* will be aces is $4/6 \times 3/5 = 12/30 = 2/5$, which is exactly the same result that we obtained previously.

In other words, if you were to bet even money in favor of a pair of aces (say one dollar per deal) on this game, then on average, for every 5 times you played, you would win twice and lose three times. Thus, on average, for every 5 plays, you would lose one dollar. Another way to look at this is that on average, you would lose $1/5$ of a dollar on every play: not a very good bet for you.

2 Roulette: A Simple Example

The following discussion is for American roulette wheels; some European wheels have 36 numbers and a “0” (zero); American wheels have the same 36 number and both a “0” and a “00” (double zero).

People place various bets on a board similar to that shown on the right in figure 1. Then the wheel (displayed on the left in the same figure) is spun and the ball bounces around and finally lands in one of the 38 slots, each of which is the same size and each of which has almost exactly the same odds of being the final resting place for the ball. Depending on the bet and where the ball stops, there are various payoffs. The 0 and 00 slots are colored green and all the others are colored red or black, half of each color.

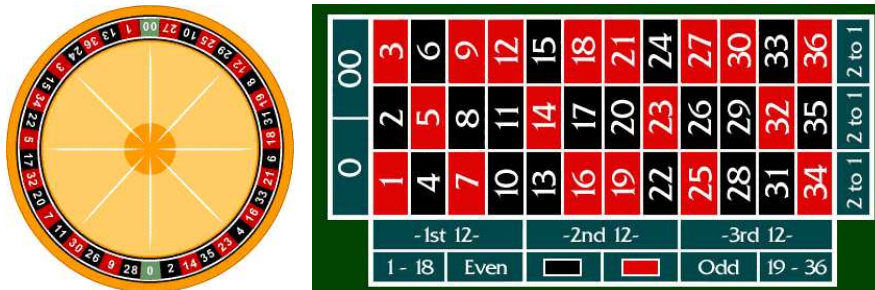


Figure 1: Roulette Board

There are many valid roulette bets, and here are just a few examples. In every case, if the number where the ball lands is not among the numbers selected, the bet is lost.

For the examples below, let us assume that one chip is bet, where the “chip” might be one dollar or 1000 euros.

1. Bet on a single number: 0, 00, or 1-36. If you win, the payoff is 35 : 1 meaning you get your original chip back plus 35 more chips. In other words, you start with 1 chip and wind up with 36 chips.
2. Bet on half the non-green numbers (all the odd or even, the red numbers or the black numbers, or the numbers 1-18 or 19-36). This pays 1 : 1, meaning you double your money if you win.
3. Bet on one-third of the numbers by placing a chip on one of the three slots labeled “2 : 1” at the bottoms of the three columns or on one of the slots labeled “1st 12”, “2nd 12” or “3rd 12”. Each winning bet pays 2 : 1.
4. Bet on two numbers adjacent on the board by placing a chip on the line between them. If this bet wins, it pays 17 : 1.

There are many more valid roulette bets but we will just consider those above. We will see that all of them give the casino (“the house”) the same advantage.

Assuming a fair, balanced wheel, the probability of each number coming up is the same: $1/38$. This means that in the long run, each number will come up, on average, 1 time in 38, so if you are betting on an individual number, on average you will win once and lose 37 times for every 38 spins of the wheel. Imagine you bet one chip on a single number each time under these circumstances. You will lose 37 chips and gain 35, on average. For every 38 plays, you “expect” to lose 2 chips, so the “expected value” to you of a single play is $-2/38$, or -5.263% .

The term “expected value” used in the previous paragraph actually has a very precise mathematical meaning that in the analysis of games of chance refers to the “average” amount one expects to win (if it’s positive) or lose (if it’s negative) for each wager. Thus, for the game of American roulette, the expected return on each bet of one unit is $-1/19$ unit: on average, you lose $1/19$ of each bet. (Of course from the casino’s point of view, the expected value of each bet is $+1/19$.)

This means that over a large number of bets at the roulette wheel, you will lose $1/19$ of the amount you bet, on average. So if you make 1000 individual \$1 bets, you will expect to lose about $\$1000/19 = \52.63 .

We only analyzed the bet on a single number, but it turns out that every roulette bet has the same expected value. Let’s consider one more, the bet on one-third of the non-green numbers. You can work out the others similarly and see that all are basically the same.

On average, for every 38 bets, you will win 12 times and lose $38 - 12 = 26$ times. Thus if you bet one chip at a time, you will lose 26 times and win 12, but since the payoff is 2 : 1, you will receive 2 chips for each win for a total of $12 \times 2 = 24$. This gives you a net loss in 38 turns of $26 - 24 = 2$ in the long run, so you lose $2/38 = 1/19$ of your money, as before.

It turns out that for games like roulette, perhaps the easiest way to find the house advantage is to imagine that you make every bet and see what happens. Suppose you

bet one chip on each of the 38 numbers. One is certain to win, and 37 are certain to lose, so you lose 37 and win 35 — a loss of 2 on every spin of the wheel.

3 Chuck-A-Luck

The casino game of chuck-a-luck consists of two cages that look something like an hourglass with three six-sided dice inside (see figure 2). The cages are spun and the dice drop from one to the other, yielding three different numbers. As was the case in roulette, you can make lots of different bets, but here we will consider only one: betting that a particular number will show up.



Figure 2: Chuck-A-Luck Cage

The payoffs are arranged so that they seem “reasonable”, or in some casinos, better than reasonable. If you bet on a number, and that number does not come up, you lose your bet. If it comes up on one die, your payoff is 1 : 1, doubling your money. If it comes up twice, your payoff is 2 : 1, but if it comes up on all three dice, you get a 10 : 1 payoff¹.

The bet, however, seems reasonable because you might think that with three dice, you’ll get your number about half the time, so you lose half the time and win half the time, but there’s the advantage of the 2 : 1 and 10 : 1 payoffs for multiple instances of your number.

The expected value of a chuck-a-luck bet on a particular number is a little harder to compute than for roulette, but it is not that difficult. We simply need to calculate the probability of obtaining one, two or three copies of that number for one particular play.

To make the calculation a little easier to follow, let’s assume that the three dice are of different colors, red, green and blue, so we can tell what happens on each die.

First, let’s calculate the chance of losing: having none of the dice display the number wagered upon. Since each die has 6 sides, the number will *not* come up five times in six, or with a probability of 5/6. The probability that all three dice will display losing numbers is simply:

$$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{125}{216}.$$

The other number that’s easy to calculate is the odds of getting all three as winners. This will happen one time in six for each die, yielding a probability that all three will

¹In some casinos the payoff for three copies of your number is 10 : 1 and in other casinos it is only 3 : 1.

be winners of:

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}.$$

To get exactly one winner, remember that the winner can be either on the red, green or blue die. It will be on the red one $1/6 \cdot 5/6 \cdot 5/6 = 25/216$ of the time. It will be on the green or blue one the same percentage, for a total probability of getting exactly one favorable number is:

$$3 \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{75}{216}.$$

The reasoning in the previous paragraph is almost the same for two winners, except that this time there are three different dice that can be a non-winner. Thus the probability of getting exactly two winners on the three dice is:

$$3 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{15}{216}.$$

We have an easy way to check the calculation: all the probabilities must add to one, since either zero, one, two or three of the dice must show the winning number. Here's the check:

$$\frac{125}{216} + \frac{75}{216} + \frac{15}{216} + \frac{1}{216} = \frac{216}{216} = 1.$$

Another way of looking at this is that for each of the six ways that the red die can land, there are six ways the green die can land for a total of $6 \cdot 6 = 36$ ways. And for each of those 36, there are six ways the blue die can land for a total of $36 \cdot 6 = 216$ ways. Arguments similar to those in the previous paragraph show that there are 125 situations with no winning numbers, 75 with one, and so on.

For every 216 rounds that we play, we will lose our chip 125 times; we'll win a chip 75 times; we'll win two chips 15 times, and we'll win ten chips once. So for an average 216 games, we will have $-125 + 75 + 2 \cdot 15 + 10 \cdot 1 = -10$. We lose ten chips every 216 plays, so our expected value in this game is $-10/216 = -4.63\%$. If the casino is one that gives you only 3 : 1 instead of 10 : 1 odds on the three-winner combination, you'll lose 17 chips every 216 times, on average, for an expected value of $-17/216 = -7.87\%$.

4 Sucker Bets

Most casino games are about like those in the previous section: the house has an advantage that's about 5 to 7 percent. That way the players don't lose money too fast, but over time, the casino makes a big profit. When players lose at a rate of 15% or more, they lose money so fast that they stop playing.

A sucker bet is usually something where you're asked to bet on something that sounds much more likely than it is. Often the expected value can amount to a large loss.

Here's a simple one, to get started. Start with a deck of cards that contains 4 aces and 2 kings. Two-thirds of the cards are aces. You're going to shuffle the deck and draw two of them. Someone is willing to bet you that there will be at least one king in those two cards.

This one is easy to work out if we calculate the odds of the reverse: the odds of getting both aces. The first card will be an ace $2/3$ of the time, since two-thirds of the cards *are* aces. But if that happens, the deck now contains three aces and two kings, so the odds of the second card also being an ace are $3/5$. Therefore the odds that both are aces are $2/3 \cdot 3/5 = 2/5$. That means you will lose 3 times out of 5. For every five times you play, on average, you will win twice and lose three times, so your expected value is $-1/5 = 20\%$. This is much worse than most casino games.

Another one is the well-known “birthday paradox”. You are in a group of 30 strangers and someone bets you that at least two of them have the same birthday. With 365 different days, this may seem very unlikely, but in fact, it is *quite* likely.

It’s similar to the previous problem in that it’s easiest to calculate the odds of *no* duplication. Making the simplifying assumptions that there are only 365 days in a year and that all birthdates are equally likely, the first person chosen eliminates one day, so the second is not a match $364/365$ of the time. The third person has to avoid two dates, so there is no match for him $363/365$ of the time, and so on. Multiplying all these together yields the following probability that there will be no matches in 30 people:

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdots \frac{336}{365} = .29368 \dots$$

Thus there will be a match seven times out of ten! Using the same technique, we find that with 50 people, only about 3 times in 100 is there no match. The break-even point is between 22 and 23 people: with 22 it is slightly more likely not to have a match, and with 23, a match occurs slightly more than half the time. At the end of this paper, in section 8, is a table showing the odds of a match with groups of various numbers of people.

5 The Monty Hall Problem

A contestant in a game show run by Monty Hall (who is a real person) is presented with three doors, two of which obscure useless prizes (usually goats in the original show) and one of which conceals a valuable prize, like a sports car. The contestant is allowed to choose a single door, but before the door is opened to reveal the contents, Monty opens one of the two doors that was not chosen, and reveals a goat behind it. At this point, the contestant is given the choice of sticking with her original choice of door, or of switching to the other door. The question is, what is the best strategy? Is it better to stick with the original door, to switch to the new door, or perhaps it doesn’t make any difference.

The argument that it shouldn’t matter is basically that since there were only two doors left, one has the goat and one the sports car, so it’s a 50 — 50 choice. I do not know of any arguments for why it would be a better idea to stick to the first choice.

The argument in favor of switching basically amounts to saying that after the first choice, the contestant has a $1/3$ chance that the prize is behind her door, and $2/3$ that it is behind one of the other two. Monty can *always* show a goat: he knows where the prize is, and must have either one or two goats behind the other doors. So seeing a goat does not change the fact that the prize is behind one of the two doors $2/3$ of the time.

Perhaps an even better argument for switching can be made if we make the problem more extreme. Suppose you've got 52 doors, which we can simulate with a deck of cards. Suppose there is one prize, the ace of spades, and none of the other cards are worth anything. You pick a card, and before looking at it, Monty turns over 50 cards, and none of them is the ace of spades. Should you switch? In this situation, it should be pretty obvious that switching is a good strategy, since after your original pick, you're pretty sure the ace is in the remaining cards. Then you're shown 50 cards that is not the ace. It's pretty clear that the remaining card is the one you want.

A somewhat related problem states that there are two cards in a hat: one is black on both sides; the other is black on one side and red on the other. The game is as follows: you draw a card from the hat and put it on the table without looking at it. When you look, the side you see that is black. What are the odds that if you turn it over, the other side will also be black?

Again, there's an argument that the odds should be 50-50: the other color is red or black, so there are only two possibilities. But as in the Monty Hall problem, things are not that simple. Just because there are only two outcomes does *not* mean that the two outcomes are equally likely. In this case, too, the odds are 2/3 that the other side is black and 1/3 that it is red. Here's why:

Call the card with two black sides "card 1" and call the one with a black and a red side "card 2". Each card has two sides: card 1 has "black 1" and "black 2"; card 2 has "red" and "black". Since you are not looking when you choose a card and place it on the table, before you look, there are four equally-likely possibilities:

card 1	black 1
card 1	black 2
card 2	black
card 2	red

Now, when you look at the situation, since you see a black card, the only possibility that can be eliminated is the fourth, and there's no reason to believe that this would make any of the remaining possibilities more or less likely. So here are the three equally-likely situations:

card 1	black 1
card 1	black 2
card 2	black

In two of those three situations (where we've got card 1 on the table) the other side will be black. In only one situation will the other side be red. This is also a good candidate for experimentation to verify that after a large number of trials, about 2/3 of the time the other side of a black card is black.

6 A Nice Classroom Example

This has worked nicely for me in a classroom, but it is *slightly* risky to you. To work out the mathematical details, you have to know how to work out combinations, and that's not covered in this article, although it's not particularly difficult.

I go into the class and tell the kids that we’re going to have a “lottery” but a much simpler one than the usual California lottery. In ours, each kid will choose 6 numbers from a set of 12 and then we will write the numbers from 1 to 12 on slips of paper, mix them in a hat, and draw 6. Any student who gets all 6 right gets \$20.

Be sure that you collect the papers with the student names and their choices of numbers *before* you run the lottery, for obvious reasons!

To figure the true odds of winning this, we assume that every set of 6 numbers is equally likely, so with 12 choices, there are:

$$\binom{12}{6} = 924$$

equally likely outcomes. That means a student would win this about one time in 924, so if there are, say, 30 kids in the class, you will lose your 20 about one time in 30.

Putting this into the “expected value” form, your expected loss in playing this game with a class of 30 kids is $(30/924) \times (\$20) = \0.649 . In other words, if you play this game over and over with a class of 30 kids, it’ll cost you, in the long run, an average of about 65 cents per game. Of course it’ll really sting on the times when you lose! If \$20 seems like too much, make the lottery so that you pick 6 numbers out of 14 or something similar; then you’ll only lose about one time in 100 for a class of 30, since:

$$\binom{14}{6} = 3003.$$

After the first game, whether anyone wins or not, ask the kids the following: “Who would pay me 25 cents to play again?” In my experience, every hand will go up.

Let’s work this out: For one kid, the expected win is $\$20/924 = \0.0216 . In other words, with no pre-payment, a player will gain about 2 cents per play. If they pay 25 to play, they will lose about 23 cents per game, on average.

7 Martingales

This doesn’t exactly fall into the “sucker bet” category, but it can make bets that are a little bit bad look extremely attractive. A martingale is basically a betting strategy that appears to make things a lot rosier than they are. Following is an example.

Suppose you’re betting on something like roulette, and every bet that you place will be for a “red” outcome. You’ll win about half the time and lose about half the time. (In fact, on an American wheel, you’ll win 9/19 of the time and lose slightly more: 10/19 of the time, but these are pretty close to a fair bet.)

Your strategy consists in making bets of different sizes, but always on the red, as follows. At any point that you win, you return to the beginning of your strategy, which is to bet a single coin on red. If you win, great—you’re one coin ahead, but if you lose, you’re one behind so you bet two. That way if you win the second time, your winnings will cover the previous one-coin loss and will give you an additional coin so that again, you’ll be one coin ahead and can go back to the beginning of the strategy.

With repeated losses, use the same method: bet exactly enough that you’ll recover all previous losses plus one additional coin so that if you win, the net progress for the

series is a gain of one coin. So if you lose twice, you'll have lost one, then two coins, so bet four. If you lose three times in a row, you've lost $1 + 2 + 4 = 7$, so bet 8. Four losses in a row amount to a total loss of $1 + 2 + 4 + 8 = 15$, so bet 16, and so on.

It's easy to show that every bet in a losing streak will be exactly double the size of the previous, so in a long series of losses, your bet sizes will be:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, . . .

The reasoning is this: You're never going to have an infinite sequence of losses, so eventually you'll win, and when that occurs, your net gain for the series is one coin. Since you can repeat this as long as you want, you've basically got an infinite money-generator.

There are two problems, of course. The first is that the bets get large, and none of us has an infinite amount of money, so as soon as the bets get to be more than we have, we will have lost a huge amount of money on that last, deadly series. And the second problem is that even if we did have fantastic quantities of money, every casino has a bet limit, and if the limit is, say, 1000 coins, if we lose at the 512 coin level, we cannot make a bet sufficient to recover our losses and the martingale fails.

Let's imagine what happens with a house limit that makes bets of over 1000 coins impossible. You'll be in trouble as soon as you lose 10 times in a row, which would happen 1 time in 1024 with a fair bet and more often with the slightly unfair bet you'll get at a casino. This will occur and cause you to lose exactly 1023 coins, so in the long run, with a fair bet, your net gain is zero.

With a slightly unfair bet (10/19 chance of a loss for roulette, for example), you'll get your 10 losses in a row about 1 time in 613, and when that happens, you'll be down 1023 coins. With this analysis, things look a lot less attractive, right?

The figures 3 and 4 show a typical plot of the player's "fortune" with time. In both cases, the house limit is set so that 128 coins at most can be bet. As expected, both graphs generally trend upward, but with catastrophic falls from time to time. The unfair martingale just has them occurring more often, and note that the general upward slopes of the long runs are slightly different. The horizontal lines in both plots show the level of the hypothetical gambler's initial fortune.

Each of the figures represents four runs with different sequences of random numbers representing the results of the trials. Both represent 8000 trials, and the unfair martingale uses the American roulette odds (10/19 chance of loss each time).

The source code in C++ that generated the figures can be found in Section 9.

8 Birthday Odds

The table below shows the probabilities of having no birthday matches in a group of n people, for $1 \leq n \leq 75$.

1	1.000000000	26	0.401759180	51	0.025568007
2	0.997260274	27	0.373140718	52	0.021995491
3	0.991795834	28	0.345538528	53	0.018861887
4	0.983644088	29	0.319031463	54	0.016123037
5	0.972864426	30	0.293683757	55	0.013737711
6	0.959537516	31	0.269545366	56	0.011667645
7	0.943764297	32	0.246652472	57	0.009877541
8	0.925664708	33	0.225028146	58	0.008335021
9	0.905376166	34	0.204683135	59	0.007010552
10	0.883051822	35	0.185616761	60	0.005877339
11	0.858858622	36	0.167817894	61	0.004911201
12	0.832975211	37	0.151265992	62	0.004090425
13	0.805589725	38	0.135932179	63	0.003395613
14	0.776897488	39	0.121780336	64	0.002809521
15	0.747098680	40	0.108768190	65	0.002316893
16	0.716395995	41	0.096848389	66	0.001904295
17	0.684992335	42	0.085969528	67	0.001559957
18	0.653088582	43	0.076077144	68	0.001273609
19	0.620881474	44	0.067114631	69	0.001036334
20	0.588561616	45	0.059024101	70	0.000840424
21	0.556311665	46	0.051747157	71	0.000679247
22	0.524304692	47	0.045225597	72	0.000547119
23	0.492702766	48	0.039402027	73	0.000439194
24	0.461655742	49	0.034220391	74	0.000351356
25	0.431300296	50	0.029626420	75	0.000280122

9 Martingale Example Source Code

```
#include <stdio.h>
#include <stdlib.h>
#include <time.h>

//double win = 0.5;
double win = 0.473684;

main(int argc, char **argv)
{
    int fortune = 0, i, bet = 1;
    double v;
    int randseed;
    struct tm *newtime;
    time_t ltime;
    time(&ltime);
    newtime = localtime(&ltime);
    randseed = newtime->tm_sec;

    printf("%%!PS-Adobe-2.0 EPSF-1.2\n");
    printf("%%Creator: Martingale\n");
    printf("%%BoundingBox: 0 0 1000 400\n");
    printf("%%EndComments\n");
    printf("0 200 translate\n");
    printf(".125 .1 scale\n");
    printf("1 setlinecap\n");
    printf("4 setlinewidth\n");
    printf("0 0 moveto 8000 0 lineto closepath stroke\n");
    printf("0 -2000 moveto 0 2000 lineto closepath stroke\n");

    unsigned short c[3] = {23, 45, 78};

    if (argc == 2) c[2] = atoi(argv[1]);
    else c[2] = randseed;
    seed48(c);
    for (i = 0; i < 8000; i++) {
        v = drand48();
        if (v < win) {
            fortune += bet; bet = 1;
        } else {
            fortune -= bet; bet = bet * 2;
            if (bet > 128) bet = 1;
        }
        printf("%d %d moveto %d %d lineto closepath stroke\n",
            i, fortune, i, fortune);
    }

    printf("showpage\n");
}
```

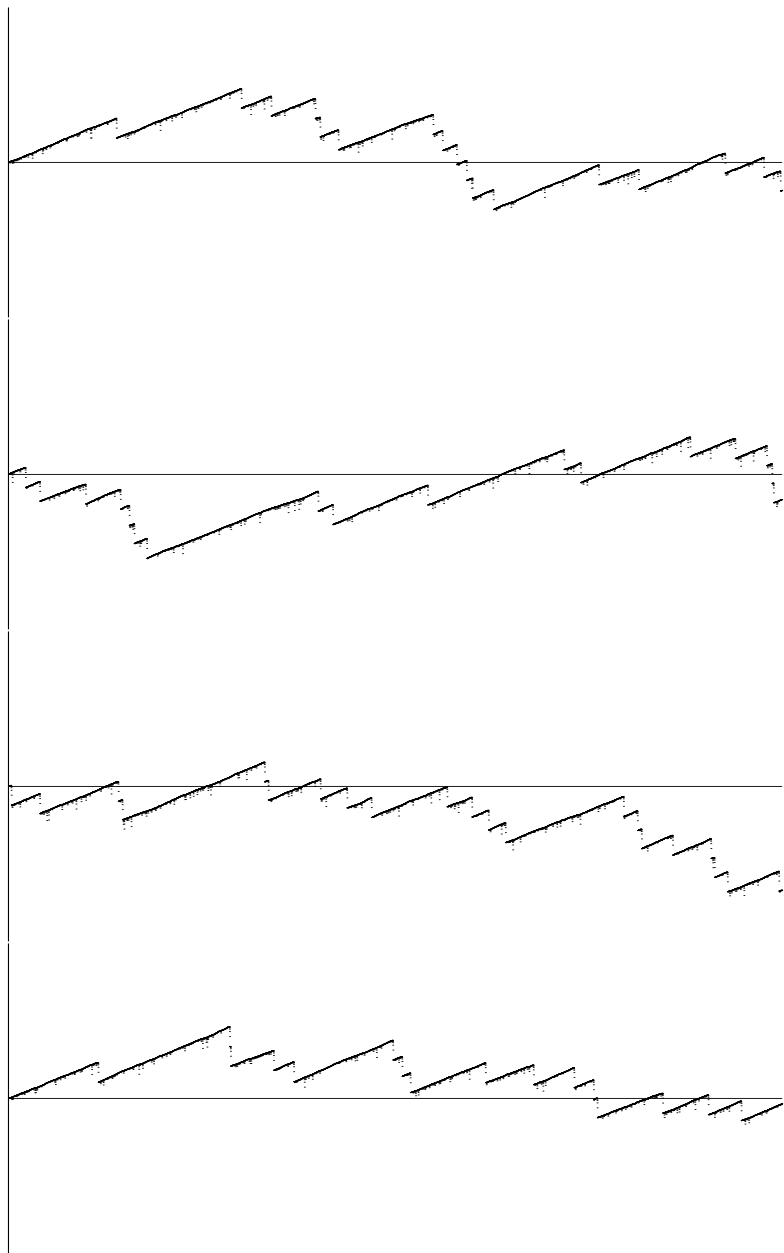


Figure 3: Fair Martingale

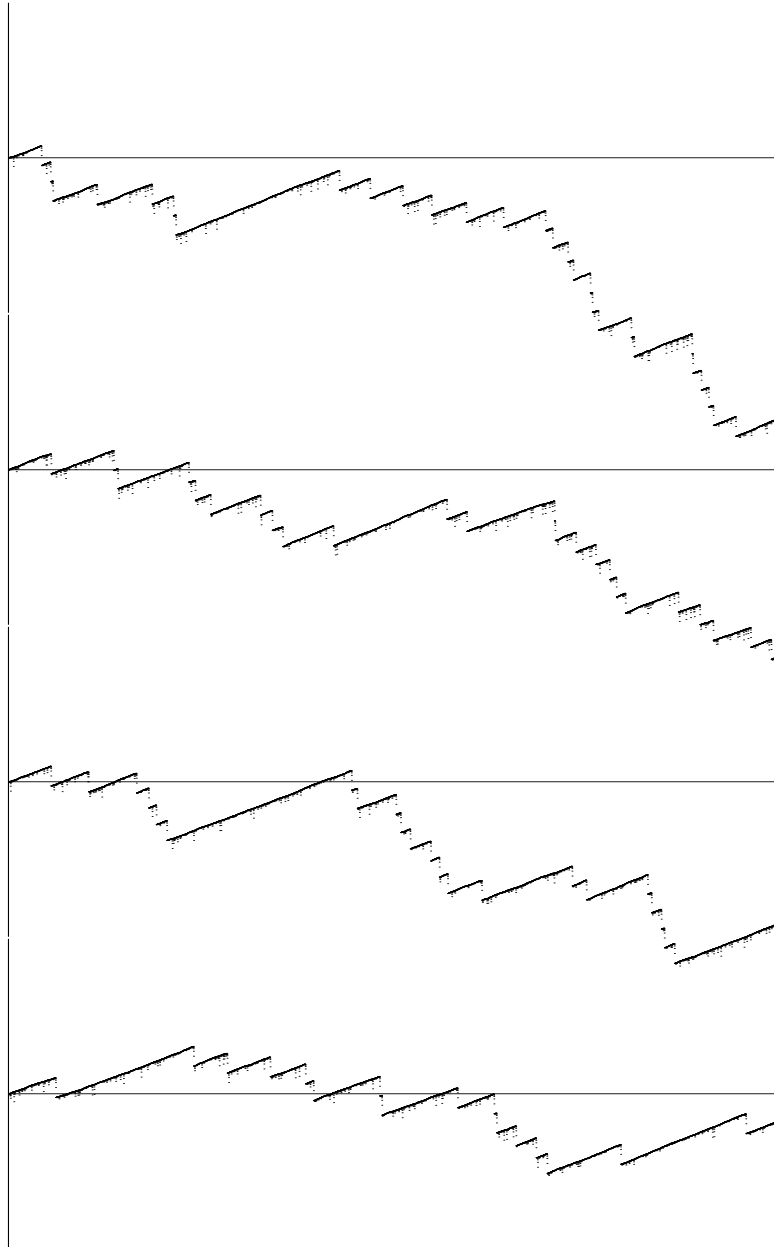


Figure 4: Unfair Martingale (9/19)