Pólya's Counting Theory

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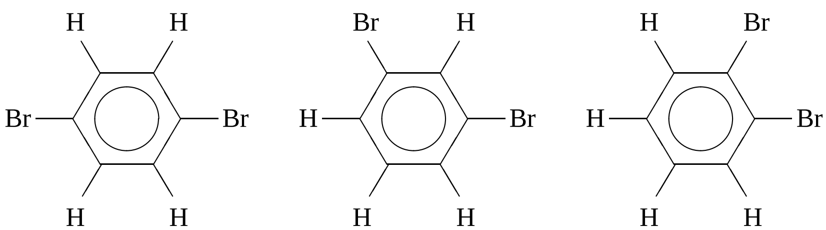
Pólya's counting theory provides a wonderful and almost magical method to solve a large variety of combinatorics problems where the number of solutions is reduced because some of them are considered to be the same as others due to some symmetry of the problem.

1 Warm-Up Problems

As a warm-up, try to work some of the following problems. The first couple are easy and then they get harder. At least read and understand all the problems before going on. Try to see the common thread that runs through them.

1. Benzene is a chemical with the formula C_6H_6. The 6 carbon atoms are arranged in a ring, and all are equivalent. If two bromine (Br) atoms are added to make a chemical with formula $C_6H_4Br_2$, there are three possible structures:



   How many structures are possible with the following formulas? In part (a) there are four hydrogen atoms, one chlorine, and one bromine atom arranged around the benzene ring; in (b), two hydrogen atoms, two chlorine atoms, and two bromine atoms; in (c), two hydrogens, an iodine, a chlorine, and two bromine atoms. The 6 carbon atoms (the C_6 part) form the benzene ring in the center. Rotating a molecule or turning it over do not turn it into a new chemical.

   (a) C_6H_4ClBr
   (b) $C_6H_2Cl_2Br_2$
   (c) $C_6H_2IClBr_2$

2. In how many ways can a strip of cloth with n stripes on it be colored with k different colors? Do not count as different patterns that are equivalent if the cloth is turned around. For example, the following two strips are equivalent, where "R" stands for "Red", "G" for "Green" and "B" for "Blue":

R	G	R	B	R	R	B

B	R	R	B	R	G	R

3. In how many ways can a tablecloth that is divided into $n \times n$ squares be colored with k colors? There are two answers, depending on whether the tablecloth can be flipped over and rotated or simply rotated to make equivalent patterns.

4. In how many ways can a necklace with 12 beads be made with 4 red beads, 3 green beads, and 5 blue beads? How many necklaces are possible with n beads of k different colors?

5. How many ways can you color the corners of a cube such that 3 are colored red, 2 are green, and 3 are blue?

6. How many ways can you color the faces of a dodecahedron with 5 different colors?

2 Illustrative Solutions

We'll begin with a few problems that are simple enough to solve without Pólya's method, which we will do, and then we will simply apply the magic method, showing the technique, but without explaining why it works, and we'll see that the same answer is obtained in both cases.

2.1 A Striped Cloth

- In how many ways can a strip of cloth with n stripes on it be colored with k different colors? Do not count as different patterns that are equivalent if the cloth is turned around. For example, the following two strips are equivalent, where "R" stands for "Red", "G" for "Green" and "B" for "Blue":

R	G	R	B	R	R	B

B	R	R	B	R	G	R

  We will consider a coloring valid if two or more adjacent stripes have the same color. In particular, a solidly-colored strip will be a perfectly good solution (where all the stripes are the same color).

If we did not consider strips to be the same when turned around, the answer is obvious—each of the n stripes can be filled with any of k colors, making a grand total of k^n possible strips. But this answer is too big, because when you turn the strip around, it matches with one that has the opposite coloring. At first it looks like we have double-counted everything, since each strip will match with its reverse, but this is obviously wrong, if we consider, say, a strip with 2 stripes and three colors. There are $3^2 = 9$ colorings (ignoring turning the strip around), but the total number of unique colorings given that we are allowed to turn the strip around is obviously not $9/2$, which is not an integer.

The problem, of course, is that some of the colorings are symmetric (in the case above, 3 of them are symmetric), so the real answer is gotten by adding the number of symmetric cases to the number of non-symmetric cases divided by 2. In this case, the calculation gives:

$$3 + \frac{9-3}{2} = 6.$$

With this in mind, the general problem is not too hard to solve; we just need to be able to count the symmetric cases. But a symmetric strip has the same stuff on the right as on the left, so once we know what's on the left, the stuff on the right is determined. There's a minor problem with odd and even sized strips, but it's not difficult. For an even number of stripes, say $n = 2m$, there are k^m different symmetric possibilities. If n is odd, $n = 2m+1$, there are k^{m+1} symmetric possibilities. Using the floor notation $\lfloor x \rfloor$ to mean the largest integer less than or equal to x, we can write this in terms of k and n as follows:

$$k^{\lfloor (n+1)/2 \rfloor}.$$

Since this is the number of symmetric colorings, the total number of colorings can be obtained with the following formula:

$$\frac{k^n - k^{\lfloor (n+1)/2 \rfloor}}{2} + k^{\lfloor (n+1)/2 \rfloor} = \frac{k^n + k^{\lfloor (n+1)/2 \rfloor}}{2}$$

We can use this formula with $n = 5$ and $k = 3$ to solve the original problem, and the answer is 135. In addition, check that the following are also true:

- If all five slots are green, clearly, there's only one way to do it.
- If the five slots must be filled with three reds and two greens, there are 6 ways to do it.
- If you can use two reds, two greens, and a blue, there are 16 ways to color it.

Now we will illustrate a method that will solve the problem (and many similar problems besides), but we will not, at first, explain how or why it works.

For what appears to be no apparent reason, look at the two permutations of the squares of the strip of cloth. Call the colored locations a, b, c, d, and e from left to right. There are two symmetry operations: leave it alone, or flip it over. In cycle notation, these correspond to: $(a)(b)(c)(d)(e)$ and $(ae)(bd)(c)$.

The first one (leave it alone) has five 1-cycles. The second (flip it over) has one 1-cycle and two 2-cycles. Let f_1 stand for 1-cycles, f_2 stand for 2-cycles. In this case there are only 1- and 2-cycles. If there were 3-cycles, we would use f_3, et cetera.

Indicate the two permutations as follows: f_1^5 and $f_1^1 f_2^2$. There is one of the first type and one of the second type, so write the following polynomial which we shall call the "cycle index":

$$P = \frac{1 \cdot f_1^5 + 1 \cdot f_1^1 f_2^2}{2}.$$

The 2 in the denominator is the total number of permutations and the 1 in front of each term in the numerator indicates that there is exactly one permutation with this structure.

Now, do the following strange "substitution". Since we're interested in three colors, we'll substitute for f_1 the term $(x + y + z)$ and for f_2, the term $(x^2 + y^2 + z^2)$. We only have f_1 and f_2 in this example, but if there were an f_5, we'd substitute $(x^5 + y^5 + z^5)$. Similarly, if there were 4 colors instead of 3, we'd use four unknowns instead of just x, y, and z.

Doing the substitution, we obtain:

$$P = \frac{(x+y+z)^5 + (x+y+z)(x^2+y^2+z^2)^2}{2}, \tag{1}$$

which, when expanded, gives:

$$\begin{aligned}10\,x\,y^3\,z + 10\,x\,y\,z^3 + 16\,x\,y^2\,z^2 + x^5 + y^5 + z^5 \\ + 16\,x^2\,y^2\,z + 10\,x^3\,y\,z + 16\,x^2\,y\,z^2 + 3\,x^4\,y + 3\,x^4\,z + 3\,x\,y^4 \\ + 3\,x\,z^4 + 6\,x^3\,y^2 + 6\,x^3\,z^2 + 6\,x^2\,y^3 + 6\,x^2\,z^3 + 3\,y\,z^4 + 3\,y^4\,z \\ + 6\,y^3\,z^2 + 6\,y^2\,z^3\end{aligned}$$

Here's the magic. If you add all the coefficients in front of all the terms: $10+10+16+\cdots+3+6+6=135$. And 135 is the total number of colorings! But there's more. The term $16xy^2z^2$ has a coefficient of 16, and that's exactly the number of ways of coloring the strip with a blue (x) two reds (y^2), and two greens (z^2). Why on earth does this work?

Actually, there is a *much* better way to "add all the coefficients"—notice that if we simply substitute 1 for x, y, and z, we get the sum of the coefficients. But there is no need to expand equation (1) before doing this—just let $x=y=z=1$ in equation (1). This gives us $(3^5+3\cdot 3^2)/2 = (243+27)/2 = 135$.

Depending on how you learn things, you may want to jump ahead to section 3.1 where we look in detail at this simple example of a striped cloth. Alternatively, you can continue reading in order to see a few more examples in detail first before looking at the underlying mathematics.

2.2 Beads on a Necklace

- Count the number of ways to arrange beads on a necklace, where there are k different colors of beads, and n total beads arranged on the necklace.

  With a necklace, we can obviously rotate it around, so if we number the beads in order as 1, 2, 3, 4, then for a tiny necklace with only four beads, the pattern "red, red, blue, blue" is clearly the same as "red, blue, blue, red", et cetera. Also, since the necklace is just made of beads, we can turn it over, so if there were four colors, although we cannot rotate "red, green, yellow, blue" into "blue, yellow, green, red", we can flip over the necklace and make those two colorings identical.

Using standard counting methods, let's solve this problem in the special case where $k=2$ and $n=4$ (two colors of beads, and only 4 beads—it's a *very* short necklace). Then we will apply Pólya's method and see that it yields the same result.

With 4 beads and two colors, we can just list the possibilities. There is obviously only one way to do it with either all red beads or all blue beads. If there is one red and three blue or the reverse—three reds and one blue, similarly, there's only one way to do it. If there are two of each, the blue beads can either be together, or can be separated, so there are two ways to do it. In total, there are thus 6 solutions.

Now let's try Pólya's method:

If the bead positions are called a, b, c, and d, here are the permutations that map the necklace into itself:

$(a)(b)(c)(d)$, $(adcb)$, $(ac)(bd)$, $(abcd)$, $(ad)(bc)$, $(ac)(b)(d)$, $(ab)(cd)$, and $(bd)(a)(c)$. (Check these.) Note that we are listing even the 1-cycles (the beads that don't move) because it will help us in setting up the equation.

In the notation we used previously, we can write down the cycle index:

$$P = \frac{1f_1^4 + 2f_1^2 f_2 + 3f_2^2 + 2f_4}{8}.$$

Since there are three permutations having two 2-cycles, there is a 3 in front of the term f_2^2, et cetera. Let $f_1 = (x + y)$, $f_2 = (x^2 + y^2)$, and $f_4 = (x^4 + y^4)$. Since there are only two colors, we only need x and y. Substitute as before to obtain:

$$P = \frac{(x+y)^4 + 2(x+y)^2(x^2+y^2) + 3(x^2+y^2)^2 + 2(x^4+y^4)}{8}. \tag{2}$$

If we expand, we obtain:

$$x^4 + x^3y + 2x^2y^2 + xy^3 + y^4.$$

It's easy to check that these terms correspond to the 6 ways beads could be arranged, where there's a unique way to do it unless there are two of each color, in which case there are two arrangements.

Also notice that it gives our detailed count as well. If we think of the x as corresponding to a "red" bead, and y to a "blue" bead, the coefficient in front of the term x^4 (which is 1) corresponds to the number of ways of making a necklace with four red beads. The 2 in front of the x^2y^2 term means that there are two necklaces with two beads of each color, et cetera.

And notice again that by substituting $x = y = 1$ into equation (2) we obtain the total count:

$$\frac{2^4 + 2\cdot 2^2 \cdot 2 + 3\cdot 2^2 + 2\cdot 2}{8} = \frac{16 + 16 + 12 + 4}{8} = 6.$$

Now let's try something slightly more interesting. What if there are three colors? Let's call the colors "R", "G" and "B", for "red", "green", and "blue".

Here's a brute-force count. Check to see that you agree with the counts below:

- All the same color (3 ways)
- Three of one color and one of another (6 ways)
- Two of one color and two of another (3 ways)
- Two of one color and two different colors (3 ways)

In the previous example (4 beads and 2 colors) we worked out how many ways there were to do all but the last one—one way with all the same color or with 3 of one color, and two ways with two of each of two colors.

A little bit of scratch work should convince you that there are also only two ways to do the last case (the two beads of the same color can be adjacent or not).

Thus the grand number of ways to place beads of 3 different colors on a necklace with 4 beads is:

$$3 \cdot 1 + 6 \cdot 1 + 3 \cdot 2 + 3 \cdot 2 = 21.$$

With three beads, the equation for the cycle index P (corresponding to equation (2) above) is:

$$P = \frac{\begin{array}{l}(x+y+z)^4 + 2(x+y+z)^2(x^2+y^2+z^2) \\ \qquad\qquad +3(x^2+y^2+z^2)^2 + 2(x^4+y^4+z^4)\end{array}}{8}. \tag{3}$$

If we just want the grand total, we can substitute $x = y = z = 1$ into equation (3) to obtain:

$$\frac{3^4 + 2 \cdot 3^2 \cdot 3 + 3 \cdot 3^2 + 2 \cdot 3}{8} = \frac{81 + 54 + 27 + 6}{8} = \frac{168}{8} = 21.$$

We can, of course, expand equation (3) and obtain:

$$\begin{array}{l}(x^4 + y^4 + z^4) + (x^3y + x^3z + xy^3 + xz^3 + y^3z + yz^3) \\ \quad +(2x^2y^2 + 2x^2z^2 + 2y^2z^2) + (2x^2yz + 2xyz^2 + 2xy^2z).\end{array}$$

Note that groups corresponding to the various combinations of beads in the list above are gathered together with parentheses.

Clearly with a small numbers of beads and colors, it's probably easier just to do a brute-force enumeration, but if the number of beads or colors gets large, Pólya's method becomes more and more attractive.

To illustrate, look at a necklace with 17 beads in it. A little playing around will show you that the cycle index polynomial you need is this:

$$P = \frac{f_1^{17} + 16f_{17} + 17f_1f_2^8}{34}.$$

Let's try to solve this with 4 colors of beads to obtain:

$$P = \frac{\begin{array}{l}(w+x+y+z)^{17} + 16(w^{17} + x^{17} + y^{17} + z^{17}) \\ \qquad +17(w+x+y+z)(w^2+x^2+y^2+z^2)^8\end{array}}{34}. \tag{4}$$

Substituting $w = x = y = z = 1$ into this yields 505421344 solutions. If you have a really strong stomach, you can multiply out the expression for P and get the breakdown for various color combinations.

Notice that if you have a particular problem, you can often solve it without a complete expansion of the expression for P. For example, if you want to know, for the 17-bead necklace, how many examples there are with 2 red, 4 blue, 3 yellow, and 8 green beads, all you need to do is to calculate the coefficient of $w^2x^4y^3z^8$ and you will have the number you want. A very valuable tool is the formula for multinomial coefficients (which is just a generalization of the formula for binomial coefficients). Here is the multinomial expansion of $(x + y)^n$, of $(x + y + z)^n$ and of $(w + x + y + z)^n$. It's easy to see what the generalization to any number of variables will be. (The binomial expansion has been written in a slightly different form than usual so you can see how it relates to the more complicated versions.)

$$\begin{aligned}(x+y)^n &= \sum_{\substack{i+j=n\\ i,j\geq 0}} \frac{n!}{i!j!}x^i y^j = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}x^i y^{n-i}\\ (x+y+z)^n &= \sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} \frac{n!}{i!j!k!}x^i y^j z^k\\ (w+x+y+z)^n &= \sum_{\substack{i+j+k+l=n\\ i,j,k,l\geq 0}} \frac{n!}{i!j!k!l!}w^i x^j y^k z^l.\end{aligned}$$

To illustrate with our example above to count the number of necklaces with 2 red, 4 blue, 3 yellow, and 8 green beads, we look at the three terms in the numerator of equation (4). We are looking for coefficients of terms like this: $w^2x^4y^3z^8$.

In $(w+x+y+z)^{17}$, the coefficient will be $17!/(2!4!3!8!)$. There will be no appropriate terms from the expansion of $16(w^{17}+x^{17}+y^{17}+z^{17})$. From $17(w+x+y+z)(w^2+x^2+y^2+z^2)^8$, the part on the right will only generate even powers of the variables, so the only way to get the term we want is to pick y from the first term, and $w^2x^4y^2z^8$ from the second, and this will occur $8!/(1!2!1!4!)$ times. So the coefficient we are interested in is:

$$\frac{\frac{17!}{2!4!3!8!} + 17\frac{8!}{1!2!1!4!}}{34} = 901320.$$

Thus, there are 901320 ways to make such a necklace.

3 What's Going On?

The construction of the functions above is rather mysterious, so let's spend a little time looking at why it might work. We'll begin by examining some very simple cases of symmetry to see why Pólya's method works on these.

Note that none of the sections below provides a proof that the method works; each section simply provides another way to think about what is going on.

3.1 Striped Cloth Analysis

Let's examine the first problem we looked at again, where we counted colorings of a striped cloth, but we'll start from the simplest possible example—a piece of cloth with one stripe. Clearly, if there are n colors available, there are n ways to color the stripe.

There is only one permutation: (1), so the associated equation for n colors will look like this:

$$P = \frac{f_1^1}{1} = \frac{x_1+x_2+\cdots+x_n}{1}.$$

This has the n terms $x_1, x_2, \ldots x_n$.

But perhaps that is too simple; let's consider a cloth with two strips. If the colors are called $A, B, C, \cdots$, then here are the possibilities for $1, 2, 3,$ and 4 colors, where $[AC]$, for example, represents the cloth with one stripe colored A and the other colored C. Of course $[AC]$ is the same as $[CA]$:

$$
\begin{aligned}
1: &\quad [AA]; && (5)\\
2: &\quad [AA][BB]; [AB] && (6)\\
3: &\quad [AA][BB][CC]; [AB][AC][BC] && (7)\\
4: &\quad [AA][BB][CC][DD]; [AB][AC][AD][BC][BD][CD] && (8)
\end{aligned}
$$

Notice that if there are n colors, there will be $\binom{n}{2}$ strips of the form $[AB]$, where A and B are different colors and n of the form $[AA]$. Thus the total number of colorings is $\binom{n}{2} + n$.

Going back to the formulation in terms of permutations, there are only two of them: $(a)(b)$ and (ab). The formula for P is:

$$P = \frac{f_1^2 + f_2}{2}.$$

Doing the magic substitution for n colors gives:

$$
\begin{aligned}
P &= \frac{(x_1 + x_2 + \cdots + x_n)^2 + (x_1^2 + x_2^2 + \cdots + x_n^2)}{2} && (9)\\
&= (x_1^2 + x_2^2 + \cdots + x_n^2) + \sum_{1 \le i < j \le n} x_i x_j. && (10)
\end{aligned}
$$

There are $\binom{n}{2}$ terms of the form $x_i x_j$ and n terms of the form x_i^2 (which you can think of as $x_i x_i$ for the benefit of a comparison to the lists in (5) through (8), above).

Every symmetry of the strip reduces the number of possible patterns. The inclusion of the permutation (ab) makes patterns $[AB]$ and $[BA]$ equivalent, where they would be considered different without that symmetry.

Consider what happens to the equation for P as additional symmetries are added (and the number of distinct colorings decreases). The denominator of P is increased by 1 for each new permutation and although new terms are added to the numerator, there are fewer of them. Consider the difference between the two terms in the numerator for P in equation (9)—the first expression, $(x_1 + \cdots + x_n)^2$ makes n^2 terms (counting multiplicity). The second expression, $(x_1^2 + \cdots + x_n^2)$, only adds n terms.

In fact, the more things a permutation moves around, the fewer terms it generates in the numerator. Let's count the terms for all the possible shapes of permutations of 4 items with n colors:

Count	Shape	Formula	Terms
1	$(a)(b)(c)(d)$	f_1^4	n^4
6	$(ab)(c)(d)$	$f_1^2 f_2$	n^3
3	$(ab)(cd)$	f_2^2	n^2
8	$(abc)(d)$	$f_1 f_3$	n^2
6	$(abcd)$	f_4	n

The count is the number of permutations having that shape; the formula is what goes in the numerator of P, and the number of terms is counted with multiplicity—in other words, x_1x_2 is counted differently from x_2x_1. Thus the permutation that doesn't collapse any colorings, $(a)(b)(c)(d)$, adds the most terms to the numerator, n^4. The permutation that moves every color position to another, $(abcd)$, has the fewest, n. Even though $(ab)(cd)$ moves everything, it moves them in a restricted way—the colors in slots a and b cannot mix into slots c and d under this permutation, and hence, since it does less collapsing, it adds more to the numerator of P (n^2 terms).

3.2 A Fixed Point

As a second example, let's consider a situation where the allowable symmetries always leave one region fixed. In the example of the strip of cloth that we considered in section (2.1), if there are an odd number of stripes, the center stripe is fixed—it always goes to itself under any symmetry operation. Here's another example: imagine a structure built with tinker-toys with a central hub and eight hubs extending from it on sticks, as in figure 1. If you've got n different colors of hubs and you want to count the number of configurations that can be made, it's pretty clear that the central hub will always go to itself in any symmetry operation. It's quite easy to make up any number of additional examples.

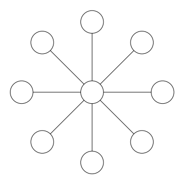


Figure 1: Tinkertoy Object

In any example where one of the positions to be colored is fixed by all of the symmetry operations, it's clear that if you can count the number of configurations of the rest of the object when n colors are used, to get the grand total when the additional fixed position is included, you'll simply multiply your previous total by n. What does this mean in terms of the permutations and the polynomial that we construct?

If you have the polynomial corresponding to the figure without the fixed point, to include the fixed point, you simply need to add a 1-cycle to each of those you already have. For example, suppose your figure consists of a triangle with the point at the center as well that is fixed. If the three vertices of the triangle are called a, b, and c, and the point at the center is called d, without the central point here are the permutations:

$$(a)(b)(c), (ab)(c), (ac)(b), (bc)(a), (abc), (acb).$$

With point d included, here they are:

$$(a)(b)(c)(d), (ab)(c)(d), (ac)(b)(d), (bc)(a)(d), (abc)(d), (acb)(d).$$

The new polynomial will simply have another f_1 in every term, so it can be factored out, and the new polynomial will simply have an additional factor of $(x_1 + x_2 + x_3 + \cdots + x_n)$ (assuming you are working the problem with n colors. The total count will thus simply be n times the previous count, as we noted above.

To make this concrete, look at this triangle case with three colors. Ignoring point d, we have:

$$P = \frac{f_1^3 + 3f_1f_2 + 2f_3}{6}.$$

If we include d, we get:

$$P = \frac{f_1^4 + 3f_1^2f_2 + 2f_1f_3}{6} = \frac{f_1(f_1^3 + 3f_1f_2 + 2f_3)}{6}.$$

If we substitute $(x+y+z)$ in the usual way, we obtain:

$$P = x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + xyz,$$

and for P' we obtain:

$$\begin{aligned} P' &= (x+y+z)P \\ &= (x+y+z)(x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + xyz). \end{aligned}$$

3.3 Independent Parts

Assume that the allowable symmetries are, in a sense, disconnected. As a simple example, imagine a child's rattle toy that has hollow balls on both ends of a handle, and one of the balls contains 3 marbles while the other contains two. In how many ways can this rattle be filled with marbles of 4 different colors? Or perhaps a more practical example is this example from chemistry: Imagine a carbon atom hooked to a nitrogen atom. You can connect three other atoms to the carbon and two other atoms to the nitrogen. If the other atoms to be hooked on are chosen from among hydrogen, fluorine, chlorine, and iodine, how many different types of chemicals are possible[1]?

In the rattle example, there is no ordering to the three marbles on one end of the rattle and to the two on the other end, but the two ends cannot be swapped since they contain different numbers of marbles. All the symmetries involve swapping among the three or among the two. So if a, b, and c represent the marbles on the three side, and if d and e represent those on the two side, we can take what's called mathematically a "direct product" of the individual groups to get the symmetry group for the entire rattle.

All the entries in the table below form the symmetry group for the rattle as a whole. Each is composed of a product of one symmetry from the three side and one from the two side:

	$(a)(b)(c)$	$(a)(bc)$	$(ab)(c)$	$(ac)(b)$	(abc)	(acb)
$(d)(e)$	$(a)(b)(c)(d)(e)$	$(a)(bc)(d)(e)$	$(ab)(c)(d)(e)$	$(ac)(b)(d)(e)$	$(abc)(d)(e)$	$(acb)(d)(e)$
(de)	$(a)(b)(c)(de)$	$(a)(bc)(de)$	$(ab)(c)(de)$	$(ac)(b)(de)$	$(abc)(de)$	$(acb)(de)$

Thus, when we have a term like f_1f_2 from the three group and an element like f_1^2 in the two group, the combination will simply generate a term that is the product of the two: $f_1^3f_2$, and this

[1] Although it may seem that these two examples are identical, they are not—the marbles in the three side can be swapped in any way (so there are 6 symmetries); on the carbon atom, they can only be rotated (so there are only 3 symmetries).

will happen in every case. It should be easy to see (if you don't see it, work out the polynomials and check) that if P_2 is the cycle index polynomial for the two group and P_3 is the one for the three group, then the cycle index polynomial for the entire permutation group will simply be P_2P_3, and it's clear that the counts of possible configurations will simply be the products of the individual configurations.

3.4 Cyclic Permutations

Next, let's look at one example that is still simple, but a bit more complicated than what we've examined up to now—we'll examine the case where the positions can be rotated by any amount, but cannot be flipped over. For concreteness, assume that you've got a circular table, and you wish to set the table with plates of k different colors, but rotations of the plates around the table are considered to be equivalent. In how many different ways can this be done?

It seems that cyclic permutations are pretty simple, but as you'll see, at least a little care must be taken. We'll examine two examples that seem similar at first, but illustrate most of the interesting behavior that you can see. We'll look at the groups of cyclic permutations of both 6 and 7 elements.

Call the positions a, b, c, d, e, and f (for the table with 6 place settings), and we'll add position g for the table with seven. Listed below are the complete sets of cyclic permutations of 6 or 7 objects.

	a	b	c	d	e	f
$(a)(b)(c)(d)(e)(f)$	a	b	c	d	e	f
$(abcdef)$	b	c	d	e	f	a
$(ace)(bdf)$	c	d	e	f	a	b
$(ad)(be)(cf)$	d	e	f	a	b	c
$(aec)(bfd)$	e	f	a	b	c	d
$(afedcb)$	f	a	b	c	d	e

	a	b	c	d	e	f	g
$(a)(b)(c)(d)(e)(f)(g)$	a	b	c	d	e	f	g
$(abcdefg)$	b	c	d	e	f	g	a
$(acegbdf)$	c	d	e	f	g	a	b
$(adgcfbe)$	d	e	f	g	a	b	c
$(aebfcgd)$	e	f	g	a	b	c	d
$(afdbgec)$	f	g	a	b	c	d	e
$(agfedcb)$	g	a	b	c	d	e	f

Notice that the lower table (for 7 elements) has every permutation except for the identity the same (in terms of cycle structure), while the table with 6 elements has a variety of cycle structures. The reason, of course, is that 7 is a prime number. With the 6-element example, three rotations of two positions or two rotations of three positions bring you back to where you started. If the plates on the 6-table are colored "red, green, red, green, red, green", they rotate to themselves after every rotation of 2 positions, or if the coloring is "red, green, blue, red, green, blue" they rotate to themselves after a rotation of three positions. In fact, it's easy to see that something similar will happen for any integer multiples of the table size. If the table, however, has a prime number of positions, the only way to bring it back to the initial configuration is to leave it alone, or turn it through an entire $360°$ rotation.

Thus if we are counting colorings that are unique even taking rotations into account, we should expect different behavior if the number of positions is prime or not. Clearly the cycle indices of the two examples above look quite different:

$$P_6 = \frac{f_1^6 + f_2^3 + 2f_3^2 + 2f_6}{6},$$

and

$$P_7 = \frac{f_1^7 + 6f_7}{7}.$$

3.5 Three More Examples

Now let's look at three related examples in detail to see exactly how the cycle structure of the symmetry permutations affect the number of possible colorings.

We will look at coloring the three points of a triangle, but with three different interpretations. We will call the three vertices of the triangle that can be colored a, b, and c:

1. No symmetry operations are allowed. In other words, coloring a red and b and c green is different from coloring b red and a and c green. The only symmetry operation allowed is to leave it unchanged. The symmetry group is this: $\{(a)(b)(c)\}$. The cycle index is this: $P_1 = (f_1^3)/1$.

2. Rotating the triangle (but not flipping it over) is allowed, So the triangle can be rotated to three different positions, or three different symmetry operations. The symmetry group is this: $\{(a)(b)(c), (abc), (acb)\}$. The cycle index is this: $P_3 = (f_1^3 + 2f_3)/3$.

3. Rotating and flipping the triangle is allowed. In this case, there are 6 symmetry operations. The symmetry group is this: $\{(a)(b)(c), (abc), (acb), (a)(bc), (b)(ac), (c)(ab)\}$. The cycle index is this: $P_6 = (f_1^3 + 3f_1f_2 + 2f_3)/6$.

In each case, let's consider the situation with three different colors allowed, so we'll be plugging in $(x^i + y^i + z^i)$ for f_i in the cycle indices above. Here's what happens in the three cases above:

1. First of all, it's clear in this case that every different assignment of colors leads to a unique coloring since only the identity symmetry operation is allowed. Thus there should be $3^3 = 27$ colorings. It's probably easiest to see what's going on by expanding $P_1 = (x + y + z)^3$ at first without using the commutative law to condense the possibilities:

$$\begin{aligned} P_1 &= xxx + xxy + xxz + xyx + xyy + xyz + xzx + xzy + xzz + \\ &\quad yxx + yxy + yxz + yyx + yyy + yyz + yzx + yzy + yzz + \\ &\quad zxx + zxy + zxz + zyx + zyy + zyz + zzx + zzy + zzz \\ &= x^3 + y^3 + z^3 + 3(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) + 6xyz \end{aligned}$$

   Before grouping like elements, the 27 terms correspond exactly to the 27 colorings, where xyy corresponds to color x in slot a, y in slot b, and y in slot c, et cetera.

   After grouping, the fact that there are 3 terms like x^2z means that there are 3 ways to color using two xs and a y.

   There's only one symmetry operation (the identity), so we only divide by 1.

2. In this case, since rotations can make certain colorings identical we'll expect to have fewer final configurations. Think of it as making all 27 colorings as in the first example, and then grouping together those that are the same.

   $$P_3 = \frac{f_1^3 + 2f_3}{3},$$

   so we're adding a bunch of terms to the numerator with the $2f_3$, but we are also dividing by 3 instead of by 1.

   Notice that the expansion of f_1^3 has more terms than any of the other possibilities—$2f_3$ in this case, and an additional $3f_1f_2$ in the following example. In this case, the $2f_3$ will contribute $2(x^3+y^3+z^3)$—just six terms (counting each one twice because of the coefficient of 2). So there are now $27 + 6 = 33$ terms in the numerator, but we divide by 3 making only 11 final terms:

   $$P_3 = x^3 + y^3 + z^3 + (x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) + 2xyz.$$

   There are 2 xyz terms since three different colors can be arranged clockwise or counter-clockwise, and since the triangle cannot be flipped over, these are distinct.

3. In this case with all possible rearrangements allowed, there should be even fewer examples (in this small case, it will only be reduced by 1, since we only will combine the two xyz terms).

   But let's look at the cycle index. The numerator will be the same as in the example above, but with the addition of the term $3f_1f_2$. The denominator will be doubled to 6.

   For three colors, $3f_1f_2 = 3(x + y + z)(x^2 + y^2 + z^2)$, which will have 27 terms (again counting multiplicity). So combining this with the 33 terms we've already considered, we have $(33 + 27)/6 = 10$ terms:

   $$P_6 = x^3 + y^3 + z^3 + (x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) + xyz.$$

It's a good idea to examine these cases yourself a bit more carefully to see exactly how the terms combine to reduce the number of colorings. You should also try to construct some other cases. For example, again with the same triangle as above, suppose the only symmetry operation allowed is to leave a in place and to swap b and c. Try it with 2 colors, or 4 colors. Try a square with various symmetry operations, et cetera.

3.6 Yet Another Approach

Let us again consider the simple case of an object that has three slots to be colored, and we will consider various symmetry operations on it and what each would do to the count of the total colorings.

3.6.1 Only the Identity

As we've said before, if no symmetry operations are allowed, there is only one symmetry operation, the identity: $(a)(b)(c)$, yielding n^3 colorings, where n is the number of available colors. $P = f_1^3 = (x_1 + \cdots + x_n)^3$ has n^3 terms so everything works out in an obvious way.

3.6.2 Identity Plus a Transposition

What if there is a single additional symmetry operation; namely, a transposition (the exchange of the first two items)? The two symmetry operations include the identity: $(a)(b)(c)$, and $(ab)(c)$.

If a coloring had different colors in slots a and b, this additional permutation will collapse those two into one. Another way of saying it is that the total number of colorings will be reduced by half the number of original colorings with different colors in the first two slots.

Originally, there were n^3 colorings. A rough approximation of the new count is $n^3/2$, but this is too small, since it also cut in half the count of colorings with identical colors in slots a and b. There are n^2 of those, so the approximation $n^3/2$ has taken out half of those n^2. Thus to get the correct count, we must add in $n^2/2$, yielding $(n^3 + n^2)/2$, which is exactly what Pólya's method gives us.

3.6.3 Identity Plus a Rotation

If we include a rotation instead of a transposition, we have $(a)(b)(c)$ and (abc). Of course if we allow rotation by one unit we can apply it twice so we have to include (acb) for a total of three symmetry operations—leave it alone, rotate one third, or rotate two thirds.

With both rotations available, the only colorings that are unaffected by them are those where all three colors are the same. With n colors, there are only n ways to color all three slots identically. Since any non-uniform coloring can undergo two rotations (or be left alone), the count of distinct non-uniform colorings is reduced by a factor of three. Thus, as above, the first approximation on the number of distinct colorings is $n^3/3$, but this has removed $2/3$ of the uniformly colored configurations, so we must add back in $2n/3$. The final count of distinct colorings is thus $(n^3 + 2n)/3$, again, exactly the result predicted by Pólya's method.

3.6.4 Full Symmetry Group

With the full symmetry group of 6 operations allowed on our three slots, the number of distinct colorings with n colors should be even smaller. The complete list of the 6 symmetries includes:

$$(a)(b)(c), (ab)(c), (ac)(b), (a)(bc), (abc), (acb).$$

Pólya's formula will be: $P = (f_1^3 + 3f_1f_2 + 2f_3)/6$. With n colors, this will give us $(n^3 + 3n^2 + 2n)/6$ distinct colorings.

With 6 symmetry group elements, each color configuration can be rearranged in 6 different ways, so the first approximation to the number of colorings is $n^3/6$. But any coloring that contains 2 colors the same and one possibly different can only be rearranged in 3 ways. There are n^2 of those, so the division of n^3 by 6 took out twice as many rearrangements as it should have, so we must add in $3n^2/6$. Finally, the count of n configurations where all three colors are the same were reduced by a factor of 6, so we've got to add $5/6$ of them back in. But we already added in $3/6$ of them when we counted configurations with at most two colors, so we need only add $2n/6$. Adding all three, we obtain:

$$\frac{n^3 + 3n^2 + 2n}{6}.$$

4 Why It Works

First let's look at a technique to solve a simple problem. Suppose you roll n dice. The total can be anything from n (if all the dice show a "1") and $6n$ (if all the dice show a "6"). There are 6^n possible outcomes if the dice are different colors so you can tell the dice apart, so what proportion of the rolls will yield a 6, 7, $\ldots 6n$?

One solution is this: To find the number of ways to obtain the sum k, simply calculate the coefficient of x^k in the expansion of the following expression:

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^n. \tag{11}$$

If you multiply this out, it is equivalent to choosing one of the terms from each of the n copies of the expression in every possible way, and so the exponent in that term in the product is the sum of the exponents of the terms that make it up.

If this isn't clear, let's first consider the specific case where $n = 2$, and instead of squaring the expression in 11 let's multiply:

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

by

$$(y^1 + y^2 + y^3 + y^4 + y^5 + y^6).$$

If we do so, we will obtain the following 36-term product:

$$\begin{array}{c} x^1y^1 + x^2y^1 + x^3y^1 + x^4y^1 + x^5y^1 + x^6y^1 \\ x^1y^2 + x^2y^2 + x^3y^2 + x^4y^2 + x^5y^2 + x^6y^2 \\ x^1y^3 + x^2y^3 + x^3y^3 + x^4y^3 + x^5y^3 + x^6y^3 \\ x^1y^4 + x^2y^4 + x^3y^4 + x^4y^4 + x^5y^4 + x^6y^4 \\ x^1y^5 + x^2y^5 + x^3y^5 + x^4y^5 + x^5y^5 + x^6y^5 \\ x^1y^6 + x^2y^6 + x^3y^6 + x^4y^6 + x^5y^6 + x^6y^6 \end{array} \tag{12}$$

Notice that all 36 combinations of one term from the first expression and one term from the second expression appear in the product. If you think of x^3 meaning "the red die's face showed a 3" and y^5 meaning "the green die's face showed a 5", then the term x^3y^5 means that "the red die's face showed a 3 *and* the green die's face showed a 5". All 36 pairs appear exactly once, corresponding exactly to the 36 different outcomes of rolling one red and one green die.

To count the number of pairs whose sum is 4, for example, we just need to count the terms whose x and y exponents add to 4. These are x^1y^3, x^2y^2 and x^3y^1, so there are 3 ways out of 36 that will show a total of 4.

If we had multiplied by an additional expression:

$$(z^1 + z^2 + z^3 + z^4 + z^5 + z^6)$$

there would be 216 different terms representing every possible combination of a red, green and blue die, where $x^1y^4z^1$ would represent the situation that the red and blue dice showed a 1 and the green die showed a 4, et cetera. Counting the number of terms in this situation that add to a particular number is a bit more labor-intensive, but the idea is simple.

But there is a much easier way to do it. If, in equation 12, we substitute x for y, then instead of having a term like x^3y^4 we would have x^3x^4, but that will simplify to make $x^{3+4} = x^7$. The 7 is the sum of the numbers on the faces of the dice. When added together, all the x^7 terms will combine, and the final coefficient of x^7 in the product will be the same as the number of pairs of dice whose faces add to 7. Here is the standard expansion of equation 11 if $n = 2$:

$$
\begin{array}{rrrrrrrrrrr}
 & & & & & x^1+ & x^2+ & x^3+ & x^4+ & x^5+ & x^6 \\
 & & & & & x^1+ & x^2+ & x^3+ & x^4+ & x^5+ & x^6 \\
\hline
 & & & & & x^7+ & x^8+ & x^9+ & x^{10}+ & x^{11}+ & x^{12} \\
 & & & & x^6+ & x^7+ & x^8+ & x^9+ & x^{10}+ & x^{11} & \\
 & & & x^5+ & x^6+ & x^7+ & x^8+ & x^9+ & x^{10} & & \\
 & & x^4+ & x^5+ & x^6+ & x^7+ & x^8+ & x^9 & & & \\
 & x^3+ & x^4+ & x^5+ & x^6+ & x^7+ & x^8 & & & & \\
x^2+ & x^3+ & x^4+ & x^5+ & x^6+ & x^7 & & & & & \\
\hline
1x^2+ & 2x^3+ & 3x^4+ & 4x^5+ & 5x^6+ & 6x^7+ & 5x^8+ & 4x^9+ & 3x^{10}+ & 2x^{11}+ & 1x^{12}
\end{array}
\tag{13}
$$

From the expansion, we can see that there is one way to obtain the total 2, two ways to obtain the total 3, and so on, up to one way to obtain the total 12.

If you raise

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)$$

to higher powers, the coefficients continue to represent the sums. For example:

$$
\begin{aligned}
(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^5 = & \\
& 1x^{30} + 5x^{29} + 15x^{28} + 35x^{27} + 70x^{26} + 126x^{25} + \\
& 205x^{24} + 305x^{23} + 420x^{22} + 540x^{21} + 651x^{20} + \\
& 735x^{19} + 780x^{18} + 780x^{17} + 735x^{16} + 651x^{15} + \\
& 540x^{14} + 420x^{13} + 305x^{12} + 205x^{11} + 126x^{10} + \\
& 70x^9 + 35x^8 + 15x^7 + 5x^6 + 1x^5
\end{aligned}
$$

so of the $6^5 = 7776$ outcomes of rolling 5 dice, 1 of them shows a sum of 30, 5 of them show a sum of 29, 15 of them show a sum of 28, and so on.

We can take advantage of the same idea of using the exponents of variables to help count configurations of the type of counting problems that we've been working on in the earlier parts of this document. For example, if we ignore symmetry altogether and ask, "How many ways are there to color a sequence of four spots with three colors?", the problem can be encoded into an expression as follows, where we think of the c_i as representing the three different colors:

$$(c_1 + c_2 + c_3)^4 = c_1c_1c_1c_1 + c_1c_1c_1c_2 + c_1c_1c_1c_3 + c_1c_1c_2c_1 + \cdots + c_3c_3c_3c_2 + c_3c_3c_3c_3.$$

There will be $3^4 = 81$ terms and they will correspond exactly to the 81 colorings of the four spots. If we simplify the terms as $c_1c_2c_3c_1 = c_1^2c_2c_3$, then after combining like terms the coefficient in front of the term $c_1^2c_2c_3$ will represent the number of such colorings that include two spots colored with the c_1 color, and one spot each of colors c_2 and c_3.

Had we been searching for combinations to color n spots with m colors, we simply need to expand:

$$(c_1 + c_2 + ... + c_m)^n$$

and combine terms as above.

Now let's look at a situation where there's a little symmetry. We'll use a special case of the example in Section 2.1. Suppose that we are using three colors on a strip of material that has five stripes, but the material can be turned around so that a "red blue green red red" is the same as a "red red green blue red" configuration. In other words, if we label the stripes from left to right as $1, 2, 3, 4, 5$, then turning the strip around is equivalent to applying the permutation $(15)(24)(3)$ to it.

If we just consider this one permutation with two 2-cycles and one 1-cycle, we can ask the following question: "How many colorings are there (using three colors) which are 'fixed' under this permutation?" In the previous sentence, 'fixed' means that the pattern looks the same after the permutation is applied. Moreover, we'd like to know how many such patterns there are using some fixed set of colors (like two blues, two reds, and a green).

It's obvious that the only patterns that will be fixed are those with stripes 1 and 5 colored the same and with stripes 2 and 4 colored the same. We don't care about the coloring of strip 3 since the permutation maps it to itself. This means that there are up to three colors to consider, some of which may be the same: the color of stripes 1 and 5, the color of stripes 2 and 4, and the color of stripe 3.

Consider the following product:

$$(c_1^2 + c_2^2 + c_3^2)^2(c_1 + c_2 + c_3) = (c_1^2 + c_2^2 + c_3^2)(c_1^2 + c_2^2 + c_3^2)(c_1 + c_2 + c_3).$$

The terms in the product are formed by choosing in every possible way one element from each set of parentheses in the expression on the right. If we think of the choice from first group as being the choice for the color of stripes 1 and 5, the choice from the second group as determining the colors of stripes 2 and 4 and the third choice as the color for the middle stripe 3, we can see that every term represents a valid coloring where the c_i terms represent the three colors. The nice thing, though, is that the c_i terms in the first two sets of parentheses are squared, indicating that the color is used twice. Thus when the terms are combined, the exponents on the c_i will indicate how many stripes of each color is required for that particular pattern.

For example, if we select the first term from the first set of parentheses, the second from the second, and the third from the third, we obtain: $c_1^2 c_2^2 c_3$ meaning that this term corresponds to one of the colorings that uses two of the first color, two of the second and one of the third.

Now imagine a more complex permutation (suppose we are coloring the four vertices of a square). Then the symmetry operation that rotates the square will look like this: (1234). The only colorings that are fixed by this permutation are those with the same color in each corner. Thus once you choose the color, you're committed to use it four times, so the term in a product where that 4-cycle appeared would look something like this:

$$(c_1^4 + c_2^4 + c_3^4),$$

assuming there are three colors.

In fact, if a cycle of length n appears in a symmetry permutation of an object to be colored, every element in that cycle must have the same color, or the object will not be fixed under that permutation.

What we have up to now is this: If there is an object with a set of positions that can be colored with k colors and we consider a permutation of those positions where the permutation has p_1 cycles of length 1, p_2 cycles of length 2, $\dots$, and p_n cycles of length n (note that many of the p_i may be zero), then if:

$$f_i = (c_1^i + c_2^i + \cdots c_k^i)$$

then if we expand and simplify:

$$f_1^{p_1} f_2^{p_2} f_3^{p_3} \cdots f_n^{p_n} \tag{14}$$

then the sum of the exponents in every term will be the same as the number of positions in the object, and the coefficient in front of each term will represent the number of distinct colorings defined by the exponent values that are fixed by the given permutation.

For a given permutation, we will call the corresponding polynomial in c_i the "pattern-inventory polynomial".

5 Burnside's Theorem

To take the final step, we'll need to learn a little bit about group theory; namely, enough to prove what is known as Burnside's theorem. We can then combine that with what we have so far to prove that Pólya's counting theory works. Before we can prove Burnside's theorem we will need to take a little detour to learn some very basic facts about group theory.

5.1 A Gentle Introduction to Group Theory

A "group" is a simple mathematical object that consists of a set G of elements (finite or infinite, but in this article we will only consider finite groups) and a single binary operation $*$ on those elements satisfying the following four conditions:

1. The operation $*$ is closed. In other words, if $a \in G$ and $b \in G$ then $a * b \in G$.
2. The operation $*$ is associative. In other words, if a, b and c are any elements of G then:

   $$a * (b * c) = (a * b) * c.$$

3. There exists an identity element $e \in G$ such that for every $a \in G$:

   $$a * e = e * a = a.$$

4. For every $a \in G$ there exists an element $a^{-1} \in G$ called the inverse of a such that:

   $$a * a^{-1} = a^{-1} * a = e,$$

   where e is the identity element mentioned above.

Note that the operation $*$ is not necessarily commutative. There may be elements $a \in G$ and $b \in G$ such that $a * b \neq b * a$.

A group can be as simple as just the identity, where $e * e = e$ is the operation, or can be terribly complicated. They can be finite or infinite, and the study of group theory is a huge subject in mathematics.

Although there are thousands of examples of groups, in this article we will only be interested in permutation groups: groups such that the elements of G are permutations of objects in some set. In such groups, if a and b are two permutations, then the permutation $a * b$ will simply be the permutation of the set's objects by applying first permutation a and next, permutation b. The identity, e, in a permutation group is simply the permutation that leaves every object where it is. For a 5-element set $\{1, 2, 3, 4, 5\}$ of objects, we would have $e = (1)(2)(3)(4)(5)$.

The inverse of a permutation a is the permutation that undoes what a does. In other words, if a moves object 1 to object 3, then a^{-1} moves object 3 to object 1. In the standard cycle notation, the inverse is obtained by reversing all the cycles. For example:

$$(1)(2\ 3\ 4)(5\ 6)(7)(8\ 9)^{-1} = (9\ 8)(7)(6\ 5)(4\ 3\ 2)(1).$$

5.1.1 S_3: The Symmetric Group on 3 Objects

Let's look in detail at a particular group:the group of all permutations of the three objects $\{1, 2, 3\}$. We know that there are $n!$ ways to rearrange n items since we can chose the final position of the first in n ways, leaving $n-1$ ways to chose the final position of the second, $n-2$ for the third, and so on. The product, $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$ is thus the total number of permutations. For three items that means there are $3! = 6$ permutations:

$$(1),\ (1\ 2),\ (1\ 3),\ (2\ 3),\ (1\ 2\ 3) \text{ and } (1\ 3\ 2).$$

Table 1 is a "multiplication table" for these six elements. Since, as we noted above, the multiplication is not necessarily commutative, the table is to be interpreted such that the first permutation in a product is chosen from the row on the top and the second from the column on the left. At the intersection of the row and column determined by these choices is the product of the permutations. For example, to multiply $(1\ 2)$ by $(1\ 3)$ choose the item in the second column and third row: $(1\ 2\ 3)$.

	(1)	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
(1)	(1)	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1\ 2)$	$(1\ 2)$	(1)	$(1\ 3\ 2)$	$(1\ 2\ 3)$	$(2\ 3)$	$(1\ 3)$
$(1\ 3)$	$(1\ 3)$	$(1\ 2\ 3)$	(1)	$(1\ 3\ 2)$	$(1\ 2)$	$(2\ 3)$
$(2\ 3)$	$(2\ 3)$	$(1\ 3\ 2)$	$(1\ 2\ 3)$	(1)	$(1\ 3)$	$(1\ 2)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3\ 2)$	(1)
$(1\ 3\ 2)$	$(1\ 3\ 2)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3)$	(1)	$(1\ 2\ 3)$

Table 1: Multiplication of permutations of 3 objects

5.1.2 Subgroups

If G is a group, we can talk about a subgroup H of the group G. We say that H is a subgroup of G if the elements of H are a subset of the elements of G, and if we restrict the operation $*$ to H, then H itself will be a group.

If we let G be the symmetric group on three objects illustrated in Section 5.1.1, we see that it has six subgroups: G itself, $\{(1)\}$, $\{(1),(1\ 2)\}$, $\{(1),(1\ 3)\}$, $\{(1),(2\ 3)\}$, and $\{(1),(1\ 2\ 3),(1\ 3\ 2)\}$. Check that all of these are subgroups. The subgroups G itself and the trivial subgroup $\{(1)\}$ are not too interesting, but they are technically subgroups.

5.1.3 Cosets

If G is a group and H is some group of G, then for any element $g \in G$, the "left coset" gH is defined to be the set:

$$gH = \{gh : h \in H\}.$$

(Note: the "right coset", Hg, is similarly defined, but we will not need it here. In the rest of this article, we will simply say "coset" instead of "left coset".)

Unless $g \in H$, the coset gH is not a group, but what is interesting is that the cosets have no overlap, they are all the same size, equal to the size of H, and that every element of G is in one of the cosets. This means that G can be divided into a number of chunks, each of which is the size of its subgroup H, which implies that the size of G is a perfect multiple of the size of H. Recall the examples in Section 5.1.2: S_3 has 6 subgroups whose sizes are: 1, 2, 2, 2, 3, and 6: all of which are divisors of 6.

Using that group as an example again, here is a list of the left cosets of $H = \{(1),(1\ 2)\}$:

$$\begin{aligned}
(1)H &= \{(1),(1\ 2)\}\\
(1\ 2)H &= \{(1),(1\ 2)\}\\
(1\ 3)H &= \{(1\ 3),(1\ 3\ 2)\}\\
(1\ 3\ 2)H &= \{(1\ 3),(1\ 3\ 2)\}\\
(2\ 3)H &= \{(2\ 3),(1\ 2\ 3)\}\\
(1\ 2\ 3)H &= \{(2\ 3),(1\ 2\ 3)\}
\end{aligned}$$

Notice that the cosets are all of size 2, which is the size of H, and that they are either identical or completely disjoint.

First, let's show that all cosets are the same size as the subgroup H. The only way that a coset can have fewer elements is if $gh_1 = gh_2$ where $h_1 \neq h_2$. But if $gh_1 = gh_2$ then $g^{-1}gh_1 = g^{-1}gh_2$ so $eh_1 = eh_2$, so $h_1 = h_2$. Thus all cosets are the same size as the subgroup H.

Obviously every element $g \in G$ is in some coset, since $g \in gH$ because $e \in H$, and $ge = g \in gH$.

Now we will show that if g_1H and g_2H share a single element g then they are identical. Let $g \in g_1H$ and $g \in g_2H$. Then $g = g_1h_1$ and $g = g_2h_2$ for some $h_1, h_2 \in H$. To show that $g_1H = g_2H$ we need to show that if $g_3 \in g_1H$, then $g_3 \in g_2H$.

If $g_3 \in g_1H$, then $g_3 = g1h_3$, for some $h_3 \in H$, and we need to show that $g_3 = g_2x$, for some $x \in H$. Since $g_1h_1 = g_2h_2$ we have $g_2 = g_1h_1h_2^{-1}$. Thus $g_2x = g_1h_1h_2^{-1}x = g_1h_3$, so if there's a solution x we will have:

$$g_1 h_1 h_2^{-1} x = g_1 h_3.$$

Then $g_1^{-1} g_1 h_1 h_2^{-1} x = g_1^{-1} g_1 h_3$, so $h_1 h_2^{-1} x = h_3$, or $x = h_2 h_1^{-1} h_3$, and since H is a group, x is clearly an element of H, so we are done.

5.1.4 Groups Acting on Sets

We are interested here in how a group G of permutations acts on a set X. As a concrete example consider first the striped cloth problem that we examined in Section 2.1. For concreteness again, let's consider a piece of cloth with 3 stripes colored with three different colors: red, green and blue. The group G consists of two permutations: one (the identity) leaves the stripes as they are and the other, $(13)(2)$, reverses them.

If we look at pieces of cloth and don't have the option of reversing them, there are 27 colorings:

RRR	RRG	RRB	RGR	RGG	RGB	RBR	RBG	RBB
GRR	GRG	GRB	GGR	GGG	GGB	GBR	GBG	GBB
BRR	BRG	BRB	BGR	BGG	BGB	BBR	BBG	BBB

These 27 colorings compose the set X and the two elements of G operate on them. The identity operates on an element of X by leaving it unchanged; the reversing permutation reverses the first and last colors, so it will map $RRB \to BRR$, $RGB \to BGR$, $RRR \to RRR$, $GBG \to GBG$, et cetera. Notice how the first permutation fixes all elements of X and the second fixes only some of them (the ones whose first and last colors are the same).

If $g \in G$ and g acts on $x \in X$ to produce $y \in X$, we write $g(x) = y$. If g is the reversing permutation in the example above, $g(RRB) = BRR$, et cetera.

Notice that we can divide the elements of the set X into "orbits" where each element in an orbit can be mapped to any other element in the orbit by some permutation in the group G. In this case the orbits are of size at most 2, since there are only 2 members of G. Here are some of the orbits for this example:

$$\{RRR\}, \{RRG, GRR\}, \{RRB, BRR\}, \{RGR\}, \{RGG, GGR\}, \ldots$$

Basically, the orbit of an element x of the set X (notation: $Orb(x)$) is the set of all elements that members of the group G can map it to:

$$Orb(x) = \{g(x) : g \in G.\}$$

If one element a is mapped to another element b in the orbit by $g \in G$, then b is mapped to a by $g^{-1} \in G$. The inverse, g^{-1}, is guaranteed to be in G since G is a group.

In the particular example above, there are 18 orbits in X under G. Make sure you understand why.

For a given $x \in X$, some members $g \in G$ fix x: $g(x) = x$, and some act on x to produce something different from x. For any particular x, an interesting subset of G (which we will prove to be a subgroup of G) is called the "stabilizer of x" (notation: $Stab(x)$):

$$Stab(x) = \{g : g \in G \text{ and } g(x) = x\}.$$

To show that $Stab(x)$ is a subgroup of G, first note that $e(x) = x$, so $e \in Stab(x)$. To show closure, if g_1 and g_2 are in $Stab(x)$, then $g_1(g_2(x)) = g_1(x) = x$, so $g_1 g_2 \in Stab(x)$. Similarly, if $g_1 \in Stab(x)$ then $g_1(x) = x$, so $g_1^{-1}(g(x)) = g_1^{-1}(x)$, so $x = g_1^{-1}(x)$, so $g_1^{-1} \in Stab(x)$.

Since $Stab(x)$ is a subgroup of G, we can consider the cosets of $Stab(x)$. If $g \in G$, then look at the coset $g\ Stab(x)$. For any $h \in Stab(x)$, $g(h(x)) = g(x)$, so every element in the coset maps x to the same place. Since the cosets are disjoint, and together include all values of $g \in G$, every coset corresponds to a different element in the orbit of x. Thus we have:

$$|G| = |Stab(x)| \cdot |Orb(x)|.$$

Since the formula above is true for every $x \in X$, we have, for every $x_i \in Orb(x)$:

$$|G| = |Stab(x_i)| \cdot |Orb(x_i)| = |Stab(x_i)| \cdot |Orb(x)|.$$

In other words, the stabilizers of each of the elements in the same orbit are the same size. (When we place the "absolute value bars" around a set or group, it means "the number of objects in". So $|G|$ is the number of objects in G.)

5.2 Statement and Proof of Burnside's Theorem

Burnside's theorem gives the relationship between the number of orbits and the number of elements fixed by particular permutations in the group G. Here is a statement of the theorem:

Theorem 1 (Burnside) *If X is a finite set and G is a group of permutations that act on X, let $F(g)$ be the number of elements of X that are fixed by a particular $g \in G$. Then the number N of orbits of X under G is given by:*

$$N = \frac{1}{|G|} \sum_{g \in G} F(g).$$

To see that it works in the particular example above, $F(e) = 27$ (e is the identity of G), and $F(g) = 9$, where g is the permutation that reverses the three colors. We have $|G| = 2$, so $(1/2)(9 + 27) = 18$, which is what we determined previously.

Consider the set $S = \{(g, x) : g \text{ fixes } x\}$. Then:

$$|S| = \sum_{g \in G} F(g),$$

since $F(g)$ is the number of elements in X that g fixes.

But another way to express the size of S is:

$$|S| = \sum_{x \in X} |Stab(x)|,$$

since the stabilizer of x contains all the elements in G that fix x.

Now, suppose that there are N orbits. Select N elements: $x_1, x_2, \ldots, x_n$ such that each one of them is in a different orbit. We then have:

$$\sum_{x \in X} |Stab(x)| = \sum_{i=1}^{N} \sum_{x \in Orb(x_i)} |Stab(x)| = \sum_{i=1}^{N} |Orb(x_i)||Stab(x_i)| = N \cdot |G|.$$

From this we have:

$$\sum_{g\in G} F(g) = |S| = \sum_{x\in X} |Stab(x)| = N \cdot |G|.$$

Divide though by $|G|$ to obtain Burnside's theorem:

$$N = \frac{1}{|G|}\sum_{g\in G} F(g).$$

5.3 Proof of Pólya's Counting Method

Recall that Pólya's method counts distinct colorings of various objects where certain colorings are considered to be equivalent if given symmetry operations map them to each other. In the earlier part of this section, the set X of items to be operated upon represented the set of all colorings where each was considered different.

When a group of symmetry operations (the group of permutations) acts on the set X, it divides the set X into a bunch of different orbits. What we want to do is count the number of orbits.

But we also noted that we can count the number of elements in the set X that are *fixed* by a given permutation by finding the pattern-inventory polynomial that corresponds to that permutation.

Each term in the pattern-inventory polynomial corresponds to a coloring with certain numbers of each color, but if you were to add all the coefficients, you would have the total number of items fixed by the permutation. This is basically the same as the function F in Burnside's theorem.

If we don't add the coefficients together, but combine all the polynomials on the right side of Burnside's theorem, we will have a polynomial, the sum of whose coefficients is the total number N of orbits, but if we look at the individual coefficients, they tell us how many orbits there are for each particular set of colors used.

At this point, with your current understanding, it is probably a good idea to go back to the beginning of this article and look over a few of the examples to make sure you see exactly what is going on in light of Burnside's theorem and Pólya's counting methods.

6 A Non-Trivial Practical Example

In this section we will consider the problem of counting the number of isomers of molecules that are derivatives of the chemical ethane (see Figure 2). Ethane is composed of two connected carbon atoms (marked with a "C" in the figure) and each of those carbon atoms is (roughly) the center of a tetrahedral arrangement of bonds. One of those bonds goes to the other carbon atom, and the other three go to other chemical groups. In the figure, positions 1, 2 and 3 are hooked to one carbon and 4, 5, and 6 are hooked to the other.

Pure ethane has a hydrogen atom at all six positions, but any or all of the positions can be replaced by other chemical groups. The molecule can rotate freely about the carbon-carbon bond. Thus if the groups at positions 1, 2 and 3 are rotated by the permutation $(1\ 2\ 3)$, the molecule is equivalent. But the direction cannot be reversed without changing the molecule: $(1\ 2)(3)$ makes a different molecule.

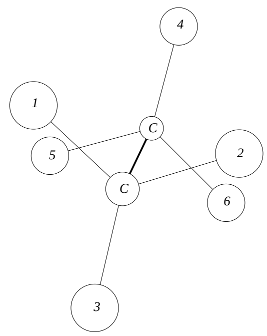


Figure 2: Ethane Compounds

If you look along the carbon-carbon bond from the end with positions 1, 2 and 3 on it, the 1, 2, 3 positions appear in clockwise order, and if you look toward the other end, the 4, 5, and 6 also appear in clockwise order, as in the figure.

So the symmetry operations that leave the molecule unchanged involve any rotation about either carbon atom, or turning the molecule around (basically exchanging positions 1 and 4, 2 and 5, and 3 and 6).

The group of permutations consists of 18 symmetries: three rotations on each end (for $3 \times 3 = 9$ of them), and then flipping the molecule's carbon atoms doubles that. The following table shows all of those symmetry permutations written in terms of the three primitive rotations (which we shall call a and b) and the carbon-carbon flip (which we shall call c). (In fact, it can be generated by just a and c or just by b and c, but the expressions (but not the permutations themselves) become more complicated.)

$e = (1)(2)(3)(4)(5)(6)$	f_1^6	$c = (1\ 4)(2\ 5)(3\ 6)$	f_2^3
$a = (1\ 2\ 3)(4)(5)(6)$	$f_1^3 f_3^1$	$ac = (1\ 5\ 2\ 6\ 3\ 4)$	f_6^1
$a^2 = (1\ 3\ 2)(4)(5)(6)$	$f_1^3 f_3^1$	$a^2c = (1\ 6\ 3\ 5\ 2\ 4)$	f_6^1
$b = (1)(2)(3)(4\ 5\ 6)$	$f_1^3 f_3^1$	$bc = (1\ 4\ 2\ 5\ 3\ 6)$	f_6^1
$b^2 = (1)(2)(3)(4\ 6\ 5)$	$f_1^3 f_3^1$	$b^2c = (1\ 4\ 3\ 6\ 2\ 5)$	f_6^1
$ab = (1\ 2\ 3)(4\ 5\ 6)$	f_3^2	$abc = (1\ 5\ 3\ 4\ 2\ 6)$	f_6^1
$ab^2 = (1\ 2\ 3)(4\ 6\ 5)$	f_3^2	$ab^2c = (1\ 5)(2\ 6)(3\ 4)$	f_2^3
$a^2b = (1\ 3\ 2)(4\ 5\ 6)$	f_3^2	$a^2bc = (1\ 6)(2\ 4)(4\ 5)$	f_2^3
$a^2b^2 = (1\ 3\ 2)(4\ 6\ 5)$	f_3^2	$a^2b^2c = (1\ 6\ 2\ 4\ 3\ 5)$	f_6^1

When we combine the 18 pattern-inventory polynomials from the table above, we obtain:

$$P = \frac{f_1^6 + 4f_1^3 f_3^1 + 4f_3^2 + 3f_2^3 + 6f_6^1}{18}. \tag{15}$$

The exponents of 1 are obviously unnecessary, but they are included to emphasize the fact that there is exactly one cycle of that length in the permutation.

If we are using n different kinds of chemical groups to be hooked to the 6 positions in the ethane-derivative molecules, then:

$$f_i = (c_1^i + c_2^i + \cdots + c_n^i),$$

where the c_j corresponds to the j^{th} chemical group.

Let's look at the situation with different numbers of groups (colors), beginning with the simplest situation: one group. Obviously, if all six positions are filled with the same group, there is only one way to do it. In Equation 15, if there is one group, $f_i = c_1^i$, yielding:

$$P = \frac{c_1^6 + 4c_1^6 + 4c_1^6 + 3c_1^6 + 6c_1^6}{18} = \frac{18c_1^6}{18} = c_1^6.$$

The coefficient 1 of c_1^6 means that there is exactly one way to construct an ethane derivative if all six attached groups are the same.

Things get a bit more interesting with two different kinds of groups:

$$\begin{aligned}
P &= \frac{(c_1+c_2)^6 + 4(c_1+c_2)^3(c_1^3+c_2^3) + 4(c_1^3+c_2^3)^2 + 3(c_1^2+c_2^2)^3 + 6(c_1^6+c_2^6)}{18} \\
&= c_1^6 + c_1^5c_2 + 2c_1^4c_2^2 + 2c_1^3c_2^3 + 2c_1^2c_2^4 + c_1c_2^5 + c_2^6 \\
&= c_1^6 + c_2^6 + \\
&\quad c_1^5c_2 + c_1c_2^5 + \\
&\quad 2c_1^4c_2^2 + 2c_1^2c_2^4 + \\
&\quad 2c_1^3c_2^3.
\end{aligned}$$

Here is the expansion with three groups:

$$\begin{aligned}
P &= \frac{\begin{array}{c}(c_1+c_2+c_3)^6 + 4(c_1+c_2+c_3)^3(c_1^3+c_2^3+c_3^3) + \\ 4(c_1^3+c_2^3+c_3^3)^2 + 3(c_1^2+c_2^2+c_3^2)^3 + 6(c_1^6+c_2^6+c_3^6)\end{array}}{18} \\
&= c_1^6 + c_1^5c_2 + c_1^5c_3 + 2c_1^4c_2^2 + 3c_1^4c_2c_3 + 2c_1^4c_3^2 + \\
&\quad 2c_1^3c_2^3 + 4c_1^3c_2^2c_3 + 4c_1^3c_2c_3^2 + 2c_1^3c_3^3 + \\
&\quad 2c_1^2c_2^4 + 4c_1^2c_2^3c_3 + 6c_1^2c_2^2c_3^2 + 4c_1^2c_2c_3^3 + 2c_1^2c_3^4 + \\
&\quad c_1c_2^5 + 3c_1c_2^4c_3 + 4c_1c_2^3c_3^2 + 4c_1c_2^2c_3^3 + 3c_1c_2c_3^4 + c_1c_3^5 + \\
&\quad c_2^6 + c_2^5c_3 + 2c_2^4c_3^2 + 2c_2^3c_3^3 + 2c_2^2c_3^4 + c_2c_3^5 + c_3^6 \\
&= c_1^6 + c_2^6 + c_3^6 + \\
&\quad c_1^5c_2 + c_1^5c_3 + c_1c_2^5 + c_1c_3^5 + c_2^5c_3 + c_2c_3^5 + \\
&\quad 2c_1^4c_2^2 + 2c_1^4c_3^2 + 2c_1^2c_2^4 + 2c_1^2c_3^4 + 2c_2^4c_3^2 + 2c_2^2c_3^4 + \\
&\quad 4c_1^3c_2^2c_3 + 4c_1^3c_2c_3^2 + 4c_1^2c_2^3c_3 + 4c_1^2c_2c_3^3 + 4c_1c_2^3c_3^2 + 4c_1c_2^2c_3^3 + \\
&\quad 3c_1^4c_2c_3 + 3c_1c_2^4c_3 + 3c_1c_2c_3^4 + \\
&\quad 2c_1^3c_2^3 + 2c_1^3c_3^3 + 2c_2^3c_3^3 + \\
&\quad 6c_1^2c_2^2c_3^2
\end{aligned}$$

In each of the final two cases above, the product is rearranged to show that the coefficients on "similar" expressions is the same. For example, in the expansion with three groups, every term that consists of four copies of one color and two of another has the same coefficient of 2: $2c_1^4c_2^2$, $2c_1^4c_3^2$, $2c_1^2c_2^4$, et cetera. This makes perfect sense, of course, since there should be the same number of ways to construct an ethane derivative with four hydrogens and two chlories as to construct one with four chlorines and two bromines.

Another thing to notice is that each expansion includes all the previous ones. If you set $c_3 = 0$ in the third one, you obtain exactly the second one, et cetera.

The examples above provide plenty of examples to check that the counting scheme works, and as exercises in counting for you. For example, with the three different groups, apparently there are six fundamentally different molecules with two of each type. Can we find them all? If the three groups are called a, b and c, how can they be divided into two groups of three to be added to each carbon atom? Here's a complete list:

$$\begin{array}{c} aab \leftrightarrow bcc \\ aac \leftrightarrow bbc \\ abb \leftrightarrow acc \\ abc \leftrightarrow abc \end{array}$$

Why are there only four? The reason is that although the first three examples have two of one and one of another on both sides, the $abc \leftrightarrow abc$ has three different types on each carbon atom, and those can be arranged clockwise or counter-clockwise as we look at the molecule along the carbon-carbon bond. So that fourth division allows three different molecule types: both arranged clockwise, both counter-clockwise, and one arranged in each direction, for a total of six, as predicted by Pólya's counting method.