Pólya's Counting Theory

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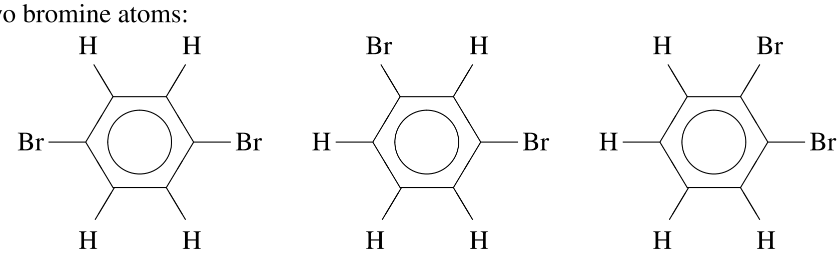
Pólya's counting theory provides a wonderful and almost magical method to solve a large variety of combinatorics problems where the number of solutions is reduced because some of them are considered to be the same as others due to some symmetry of the problem.

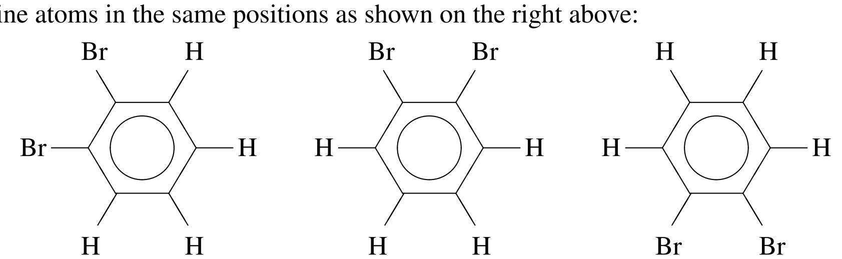
1 Warm-Up Problems

As a warm-up, try to work at least the first two of the following problems. The first two are relatively easy and then they get harder. At least read and understand all the problems before going on. Try to see the common thread that runs through them.

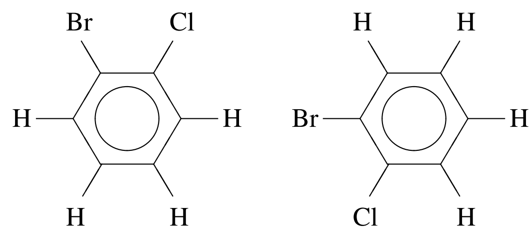
1. Benzene is a chemical with the formula C_6H_6. The 6 carbon atoms are arranged in a ring, and all are equivalent. There is a hydrogen (H) atom attached to each of the carbons. The molecule is flat, so it also looks the same if it is turned over. If two of the hydrogen atoms are replaced by bromine (Br) atoms to make a chemical with formula $C_6H_4Br_2$, there are three possible structures that correspond to different numbers of hydrogen atoms between the two bromine atoms:



   Since all the carbon positions are identical, each of these molecules could be drawn in different ways. In total there are six ways to draw the example on the right above, three of which appear below, but note that all of them can be rotated or flipped over to place the two bromine atoms in the same positions as shown on the right above:



Note that in this example, only rotations are needed to make the different pictures look the same. The two examples below which include both bromine and chlorine (Cl) atoms are equivalent, but to make the drawings the same, one of them must be flipped over (together with a possible rotation):



How many structures are possible with the following formulas? In part (a) there are four hydrogen atoms, one chlorine, and one bromine atom arranged around the benzene ring; in (b), two hydrogen atoms, two chlorine atoms, and two bromine atoms; in (c), two hydrogens, an iodine (I), a chlorine, and two bromine atoms. The 6 carbon atoms (the C_6 part) form the benzene ring in the center. Rotating a molecule or turning it over do not turn it into a new chemical.

(a) C_6H_4ClBr

(b) $C_6H_2Cl_2Br_2$

(c) $C_6H_2IClBr_2$

2. In how many ways can a long thin piece of cloth with n stripes on it be colored with k different colors? Do not count as different patterns that are equivalent if the cloth is turned around. For example, the following two colorings are equivalent, where “R” stands for “Red”, “G” for “Green” and “B” for “Blue”:

R	G	R	B	R	R	B

B	R	R	B	R	G	R

Check the solution by showing that if there are 5 stripes and 3 colors then there are 135 colorings. How many are of a solid color? How many have three reds and two greens? How many have two reds, two greens, and a blue?

3. In how many ways can a square tablecloth that is divided into 5×5 squares be colored with k colors? There are two answers, depending on whether the tablecloth can be flipped over and rotated or simply rotated to make equivalent patterns.

4. How many ways can you color the six faces of a cube such that 1 is colored red, 2 are green, and 3 are blue? How many total red-green-blue colorings of the cube are there?

5. How many ways can you color the faces of a regular dodecahedron with 5 different colors? How many ways can you color them with red and four other colors where exactly 5 of the faces are colored red and the other faces can be colored arbitrarily?

6. In how many ways can a necklace with 12 beads be made with 4 red beads, 3 green beads, and 5 blue beads? How many necklaces are possible with n beads of k different colors?

(Depending on the type of beads, some necklaces can be turned over and some cannot, so there are really four different problems here.)

Solving all of the problems above is much easier once the machinery of Pólya's method is available, and some of the problems above will be very difficult to solve without those tools. Solutions to the harder problems above, based on Pólya's method, appear in Appendix C.

2 Illustrative Solutions

We'll begin with a few problems that are simple enough to solve without Pólya's method, which we will do, and then we will simply apply the magic method, showing the technique, but without explaining why it works, and we'll see that the same answer is obtained in both cases.

2.1 A Striped Cloth

In how many ways can a strip of cloth with n stripes on it be colored with k different colors? Do not count as different patterns that are equivalent if the cloth is turned around. For example, the following two strips are equivalent, where "R" stands for "Red", "G" for "Green" and "B" for "Blue":

R	G	R	B	R	R	B

B	R	R	B	R	G	R

Valid colorings include situations where two or more adjacent stripes have the same color. In particular, a solidly-colored strip will be a perfectly good solution (where all the stripes are the same color).

If we did not consider strips to be the same when turned around, the answer is obvious—each of the n stripes can be filled with any of k colors, making a grand total of k^n possible strips. But this answer is too big, because when you turn the strip around, it matches with one that has the opposite coloring. At first it looks like we have double-counted everything, since each strip will match with its reverse, but this is obviously wrong. Consider, a strip with 2 stripes and three colors. There are $3^2 = 9$ colorings (ignoring turning the strip around), but the total number of unique colorings (when we are allowed to turn the strip around) is obviously not $9/2$, which is not an integer.

The problem, of course, is that some of the colorings are symmetric (in the case above, 3 of them are symmetric), so the real answer is gotten by adding the number of symmetric cases to the number of non-symmetric cases divided by 2. In this case, the calculation gives:

$$3 + \frac{9-3}{2} = 6.$$

With this in mind, the general problem is not too hard to solve; we just need to be able to count the symmetric cases. A symmetric strip has the same colors on the right as on the left, so once we know what's on the left, the colors on the right are determined. There's a minor problem with odd and even sized strips, but it's not difficult. For an even number of stripes, say $n = 2m$, there are k^m different symmetric possibilities. If n is odd, $n = 2m + 1$, there are k^{m+1} symmetric possibilities. Using the floor notation $\lfloor x \rfloor$ to mean the largest integer less than or equal to x, we can write this in terms of k and n as follows:

$$k^{\lfloor (n+1)/2 \rfloor}.$$

Since this is the number of symmetric colorings, the total number of colorings can be obtained with the following formula:

$$\frac{k^n - k^{\lfloor (n+1)/2 \rfloor}}{2} + k^{\lfloor (n+1)/2 \rfloor} = \frac{k^n + k^{\lfloor (n+1)/2 \rfloor}}{2}$$

We can use this formula with $n = 5$ and $k = 3$ to solve the original problem, and the answer is 135. In addition, check that the following are also true:

- If all five slots are green, clearly, there's only one way to do it, so there are three different solidly-colored strips.
- If the five slots must be filled with three reds and two greens, there are 6 ways to do it.
- If you can use two reds, two greens, and a blue, there are 16 ways to color it.

Now we will illustrate a method that will solve the problem (and many similar problems besides), but we will not, at first, explain how or why it works. The following method requires that you understand how to manipulate permutations. If you do not, Appendix A provides a gentle introduction to them.

For what appears to be no apparent reason, look at the two permutations of the squares of the strip of cloth. Call the colored locations 1, 2, 3, 4, and 5 from left to right. There are two symmetry operations: leave it alone, or flip it over. In cycle notation, these correspond to: $(1)(2)(3)(4)(5)$ and $(3)(15)(24)$.

The first one (don't rearrange anythinng) has five 1-cycles. The second (reverse it) has one 1-cycle and two 2-cycles. Let f_1 stand for 1-cycles, f_2 stand for 2-cycles. In this case there are only 1- and 2-cycles. If there were 3-cycles, we would use f_3, et cetera. Ignoring the actual content of the cycles, simply replace every 1-cycle with an f_1, every 2-cycle with an f_2, et cetera.

With this substitution the two permutations are converted to: f_1^5 and $f_1^1 f_2^2$. There is one of the first type and one of the second type, and we write the following polynomial which we shall call the "cycle index":

$$P = \frac{1 \cdot f_1^5 + 1 \cdot f_1^1 f_2^2}{2}.$$

The 2 in the denominator is the total number of permutations and the 1 in front of each term in the numerator indicates that there is exactly one permutation with this structure.

Now, do the following strange "substitution". Since we're interested in three colors, we'll substitute for f_1 the term $(x + y + z)$ and for f_2, the term $(x^2 + y^2 + z^2)$. We only have f_1 and f_2 in this example, but if there were an f_5, for example, we'd substitute $(x^5 + y^5 + z^5)$. Similarly, if there were 4 colors instead of 3, we would use four unknowns instead of three: w, x, y, and z.

Doing the substitution with three variables we obtain:

$$P = \frac{(x+y+z)^5 + (x+y+z)(x^2+y^2+z^2)^2}{2}, \tag{1}$$

which, when expanded (which involves a lot of algebra), gives:

$$\begin{aligned}
&10\,x\,y^3\,z + 10\,x\,y\,z^3 + 16\,x\,y^2\,z^2 + x^5 + y^5 + z^5 \\
&\quad + 16\,x^2\,y^2\,z + 10\,x^3\,y\,z + 16\,x^2\,y\,z^2 + 3\,x^4\,y + 3\,x^4\,z + 3\,x\,y^4 \\
&\quad + 3\,x\,z^4 + 6\,x^3\,y^2 + 6\,x^3\,z^2 + 6\,x^2\,y^3 + 6\,x^2\,z^3 + 3\,y\,z^4 + 3\,y^4\,z \\
&\quad + 6\,y^3\,z^2 + 6\,y^2\,z^3
\end{aligned}$$

Here's the magic. If you add all the coefficients in front of all the terms: $10 + 10 + 16 + 1 + \cdots + 3 + 6 + 6 = 135$. And 135 is the total number of colorings! But there's more. The term $16xy^2z^2$ has a coefficient of 16, and that's exactly the number of ways of coloring the strip with one blue (x), two reds (y^2), and two greens (z^2). Pretty amazing, no?

Actually, there is a *much* better way to "add all the coefficients"—notice that if we simply substitute 1 for x, y, and z, we get the sum of the coefficients. But there is no need to expand equation (1) before doing this—just let $x = y = z = 1$ in equation (1). This gives us:

$$\frac{3^5 + 3 \cdot 3^2}{2} = \frac{243 + 27}{2} = 135.$$

2.2 Beads on a Necklace

Count the number of ways to arrange beads on a necklace, where there are k different colors of beads, and n total beads arranged on the necklace.

With a necklace, we can obviously rotate it around, so if we number the bead positions in order as 1, 2, 3, 4, then for a tiny necklace with only four beads, the pattern "red, red, blue, blue" is clearly the same as "red, blue, blue, red", et cetera. Also, since the necklace is just made of beads, we can turn it over, so if there were four colors, although we cannot rotate "red, green, yellow, blue" into "blue, yellow, green, red", we can flip over the necklace and make those two colorings identical.

Note: Counting the number of molecules obtained by substituting different atoms connected to the six carbons of a benzene ring that we studied in the first section of this document is mathematically exactly the same as a six-bead necklace.

Using standard counting methods, let's solve this problem in the special case where $k = 2$ and $n = 4$ (two colors of beads, and only 4 beads—it's a *very* short necklace). Then we will apply Pólya's method and see that it yields the same result.

With four beads and two colors (say red and blue), we can just list the possibilities. There is obviously only one way to do it with either all red beads or all blue beads. If there is one red and three blue or the reverse—three reds and one blue, similarly, there's only one way to do it. If there are two of each, the blue beads can either be together, or can be separated, so there are two ways to do it. In total, there are thus 6 solutions.

Now let's try Pólya's method:

If the bead positions (slots for colors) are called 1, 2, 3, and 4, here are the permutations that map the necklace into itself. Four are just rotations of the necklace and four are obtained by flipping it over (and possibly rotating the flipped necklace):

(1)(2)(3)(4), (1 4 2 3), (1 3)(2 4), (1 2 3 4), (1 4)(2 3), (2)(4)(1 3), (1 2)(3 4), and (1)(3)(2 4). (Check these.) Note that we are listing even the 1-cycles (the beads that don't move) because it will help us in setting up the equation.

In the notation we used previously, we can write down the cycle index:

$$P = \frac{1f_1^4 + 2f_1^2 f_2 + 3f_2^2 + 2f_4}{8}.$$

The numerator will have 8 terms corresponding to each of the 8 permutations but since there are three permutations having two 2-cycles, there is a 3 in front of the term f_2^2, et cetera. Since there are only two colors our substitutions will have only terms in x and y. Let $f_1 = (x + y)$, $f_2 = (x^2 + y^2)$, and $f_4 = (x^4 + y^4)$. Substitute as before to obtain:

$$P = \frac{(x+y)^4 + 2(x+y)^2(x^2+y^2) + 3(x^2+y^2)^2 + 2(x^4+y^4)}{8}. \quad (2)$$

If we expand, we obtain:

$$x^4 + x^3y + 2x^2y^2 + xy^3 + y^4.$$

It's easy to check that these terms correspond to the 6 ways beads could be arranged. When there are two of each color there are two ways to do it which is why there is a 2 in front of the x^2y^2 term. In every other case, there is only one way to fill the necklace so each of the other terms effectively has a coefficient of 1.

Also notice that it gives our detailed count as well. If we think of the x as corresponding to a "red" bead, and y to a "blue" bead, the coefficient in front of the term x^4 (which is 1) corresponds to the number of ways of making a necklace with four red beads. The 2 in front of the x^2y^2 term means that there are two necklaces with two beads of each color, et cetera.

And notice again that by substituting $x = y = 1$ into equation (2) we obtain the total count:

$$\frac{2^4 + 2 \cdot 2^2 \cdot 2 + 3 \cdot 2^2 + 2 \cdot 2}{8} = \frac{16 + 16 + 12 + 4}{8} = 6.$$

Now let's try something slightly more interesting. What if there are three colors? Let's call the colors "R", "G" and "B", for "red", "green", and "blue".

Here's a brute-force count. Check to see that you agree with the counts below:

- All the same color (3 ways). There are three colors.
- Three of one color and one of another (6 ways). There are three ways to pick a pair of colors and for each of those, two ways to choose which color appears three times.
- Two of one color and two of another (6 ways). Again, there are three ways to choose the color pair, and the colors can be either together or split in the necklace

- Two of one color and two different colors (6 ways). There are three ways to choose the color that is duplicated, and the other two colors can be either together on the necklace or split by the duplicated color.

Altogether there are $3 + 6 + 6 + 6 = 21$ ways to do it.

With three beads, the equation for the cycle index P (corresponding to equation (2) above) is:

$$P = \frac{\begin{array}{l}(x+y+z)^4 + 2(x+y+z)^2(x^2+y^2+z^2) \\ \qquad\qquad +3(x^2+y^2+z^2)^2 + 2(x^4+y^4+z^4)\end{array}}{8}. \tag{3}$$

If we just want the grand total, we can substitute $x = y = z = 1$ into equation (3) to obtain:

$$\frac{3^4 + 2\cdot 3^2 \cdot 3 + 3\cdot 3^2 + 2\cdot 3}{8} = \frac{81 + 54 + 27 + 6}{8} = \frac{168}{8} = 21.$$

We can, of course, expand equation (3) and obtain:

$$\begin{array}{l}(x^4 + y^4 + z^4) + (x^3y + x^3z + xy^3 + xz^3 + y^3z + yz^3) \\ \qquad +(2x^2y^2 + 2x^2z^2 + 2y^2z^2) + (2x^2yz + 2xyz^2 + 2xy^2z).\end{array}$$

Note that groups corresponding to the various combinations of beads in the list above are gathered together with parentheses.

Clearly with a small numbers of beads and colors, it's probably easier just to do a brute-force enumeration, but if the number of beads or colors gets large, Pólya's method becomes more and more attractive.

To illustrate, look at a necklace with 17 beads in it. The number 17 is prime so if the necklace is not flipped over every rotation (except for a rotation by zero) creates a cycle of length 17 since if it created cycles of, say, length 3 then 17 would have to be a multiple of 3. To see an example, let's look at the cycle structure generated by advancing each slot by three positions: slot 1 goes to slot 4, and so on. Here's what we get:

$$(1\ 4\ 7\ 10\ 13\ 16\ 2\ 5\ 8\ 11\ 14\ 17\ 3\ 6\ 9\ 12\ 15).$$

If it is not obvious, try some other length of advancement and see that a similar thing happens. (For more information on this idea, check out Section 3.6.)

Thus without flipping, there is one cycle index term of the form f_1^{17} and 16 of the form f_{17}^1.

If it is flipped it can be flipped over an axis passing through one bead and the center of the necklace. This will leave one bead fixed and swap the other 8 pairs. Since this can be done across any of the 17 beads, there will be 17 terms of the form $f_1^1 f_2^8$.

Thus the cycle index polynomial is this:

$$P = \frac{f_1^{17} + 16f_{17} + 17f_1f_2^8}{34}.$$

If we want to work out the details with 4 colors of beads we will need to work with:

$$P = \frac{\begin{array}{c}(w+x+y+z)^{17} + 16(w^{17}+x^{17}+y^{17}+z^{17}) \\ +17(w+x+y+z)(w^2+x^2+y^2+z^2)^8\end{array}}{34}. \quad (4)$$

Substituting $w = x = y = z = 1$ into this yields 505421344 solutions. If you have a really strong stomach, you can expand the expression for P and get the breakdown for various color combinations.

Notice that if you have a particular problem, you can often solve it without a complete expansion of the expression for P. For example, if you want to know, for the 17-bead necklace, how many examples there are with 2 red, 4 blue, 3 yellow, and 8 green beads, all you need to do is to calculate the coefficient of $w^2x^4y^3z^8$ and you will have the number you want.

2.3 Multinomial Coefficients

A very valuable tool is the formula for multinomial coefficients (which is just a generalization of the formula for binomial coefficients). Here are the multinomial expansions of $(x+y)^n$, of $(x+y+z)^n$, and of $(w+x+y+z)^n$. It's easy to see what the generalization to any number of variables will be. (The binomial expansion has been written in a slightly different form than usual so you can see how it relates to the more complicated versions.)

$$\begin{aligned}
(x+y)^n &= \sum_{\substack{i+j=n \\ i,j\geq 0}} \frac{n!}{i!j!}x^iy^j = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}x^iy^{n-i} \\
(x+y+z)^n &= \sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} \frac{n!}{i!j!k!}x^iy^jz^k \\
(w+x+y+z)^n &= \sum_{\substack{i+j+k+l=n \\ i,j,k,l\geq 0}} \frac{n!}{i!j!k!l!}w^ix^jy^kz^l.
\end{aligned}$$

Note: You may not have seen the notation above where there are multiple (in this case, two) conditions below the "Sigma" symbol. That means the sum is taken over all possible combinations of the variables that satisfy all the conditions. For example the second formula indicates that the sum is to be taken over all triples of integers i, j, and k where all three are non-negative and such that the sum of the three is n.

To illustrate with our example above to count the number of necklaces with 2 red, 4 blue, 3 yellow, and 8 green beads, we look at the three terms in the numerator of equation (4). We are looking for coefficients of terms like this: $w^2x^4y^3z^8$.

In $(w+x+y+z)^{17}$, the coefficient will be $17!/(2!4!3!8!)$ simply by looking at the appropriate multinomial coefficient. There will be no appropriate terms from the expansion of $16(w^{17}+x^{17}+y^{17}+z^{17})$. From $17(w+x+y+z)(w^2+x^2+y^2+z^2)^8$, the part on the right will only generate even powers of the variables, so the only way to get the term we want is to pick y from the first term, and $w^2x^4y^2z^8$ from the second, and this will occur $8!/(1!2!1!4!)$ times. So the coefficient we are interested in is:

$$\frac{\frac{17!}{2!4!3!8!}+17\frac{8!}{1!2!1!4!}}{34}=901320.$$

Thus, there are 901320 ways to make such a necklace.

3 Motivation for Pólya's Method

The construction of the functions above is rather mysterious, so let's spend a little time looking at why it might work. We'll begin by examining some very simple cases of symmetry to see why Pólya's method works on these.

Note that none of the sections below provides a proof that the method works; each section simply provides another way to think about what is going on and to convince you that there are good intuitive reasons to believe that Pólya's method works.

For an actual proof of the validity of the method, you can skip ahead to Section 4 and continue from there.

3.1 Groups as Operators

We have been careful to describe the group of symmetries as operations that move the objects in the slots. The "objects" may be atoms or colors or anything else. Since is doesn't matter mathematically what is in the slots, in this section we will always just say they are colors. A group operation may say something like "Swap the colors in slots 1 and 2", but if colors in those slots happen to be the same, then applying the group operation will not make any change in the resulting coloration.

In fact if you have a situation with n slots and k colors then before you start looking at the effects of the symmetry operations, every slot can be assigned any color yielding n^k possible colorations. When you look at the symmetries some of those colorations will be indistinguishable, and so the number of different ones will be less than the n^k maximum possible colorings.

In fact, with each additional symmetry more of the colorings may be lumped together. Ideally, at the end of the analysis is a list of the groups of colorings that are equivalent because of the symmetry operations.

We will present a mathematically-precise description of how groups behave as operators in Section 4.1

3.2 No Symmetry (Well, Actually One Symmetry)

There is always one symmetry; namely, the one that does nothing, leaving every slot exactly as it was. If there are n slots, the "do nothing" symmetry looks like this:

$$(1)(2)(3)\cdots(n)$$

so the cycle index will be:

$$f_1^n/1.$$

If there are k colors and the variables are $x_1, x_2, \ldots, x_k$ then the polynomial will be:

$$(x_1 + x_2 + \cdots + x_k)^n$$

whose coefficients will sum to k^n and the coefficients in front of the individual terms will count the number of ways to fill the slots with a particular set of colors.

For example, if $k = 3$ and $n = 4$ (and using x, y, and z as the three variable names) we obtain:

$$P = (x+y+z)^4 = \begin{array}{r} (x^4+y^4+z^4) + 4(x^3y + xy^3 + x^3z + xz^3 + y^3z + zy^3) + \\ 6(x^2y^2 + x^2z^2 + y^2z^2) + 12(xyz^2 + xy^2z + x^2yz). \end{array}$$

The interpretation is that there is one way to do the coloring with a single color:

$$xxxx$$

there are four ways to have three of one color and one of another:

$$xxxy, xxyx, xyxx, yxxx$$

six ways to have two of one color and two of another:

$$xxyy, xyxy, xyyx, yxxy, yxyx, yyxx$$

and finally, twelve ways to have two of one color and one each of the others:

$$xxyz, xxzy, xyxz, xzxy, xyzx, xzyz, yxxz, zxxy, yxzx, zxyx, yzxx, zyxx.$$

Of course these numbers are just the multinomial coefficients described in Section 2.3.

3.3 One Color

Another trivial situation to examine is when a single color is used on (possibly complex) set of symmetries of an object. The answer had better be that there is only one way to do it.

Remember that every permutation of n slots in the symmetry is made of a set of cycles such that every slot is mentioned in exactly one of the cycles (which may be a cycle of length 1). The term corresponding to that permutation in the cycle index will look like this:

$$f_1^{p_1} f_2^{p_2} f_3^{p_3} \cdots f_n^{p_n}.$$

Any number of the p_i may be zero (if there are no cycles of length i) but since the sum of the lengths of the cycles is n it has to be the case that:

$$1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + \cdots + n \cdot p_n = n.$$

Since there is only one color, the substitution into any f_1 terms is x. Into f_2 terms, x^2. Into f_3 terms, x^3, et cetera. In the product, the exponent on x will be exactly the same as in the sum above;

namely, n, so for every permutation one term of the form x^n will be added. Since there are, say, m permutations, there will be m copies of x^n in the numerator and since there are m permutations, the denominator of the cycle index will be m, so the cycle index will be:

$$m \cdot x^n/m = x^n.$$

Pólya's interpretation of this is that there is exactly one coloring where all n slots are filled with the same color and that is exactly what we expect it to be.

3.4 Fixed Slots

Consider a situation where all the allowable symmetries leave one (or more) slots fixed. In the example of the strip of cloth that we considered in section (2.1), if there are an odd number of stripes, the center stripe is fixed—it always goes to itself under any symmetry operation. Here's another example: imagine a structure built with tinker-toys with a central hub and eight hubs extending from it on sticks, as in figure 1. If you've got n different colors of hubs and you want to count the number of configurations that can be made with some number of colors, it's pretty clear that the central hub will always go to itself in any symmetry operation. It's quite easy to make up any number of additional examples.

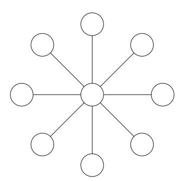


Figure 1: Tinkertoy Object

In any example where one of the positions to be colored is fixed by all of the symmetry operations, it's clear that if you can count the number of configurations of the rest of the object when n colors are used, to get the grand total when the additional fixed position is included, you'll simply multiply your previous total by n. What does this mean in terms of the permutations and the polynomial that we construct?

If you have the polynomial corresponding to the figure without the fixed point, to include the fixed point, you simply need to add a 1-cycle to each of those you already have. For example, suppose your figure consists of a triangle with the point at the center as well that is fixed. If the three vertices (the slots) of the triangle are called 1, 2, and 3, and the point at the center is called 4, without the central point here are the permutations:

$$(1)(2)(3), (3)(12), (2)(13), (1)(23), (123), (132).$$

With point 4 included, here they are:

$$(1)(2)(3)(4), (3)(4)(12), (2)(4)(13), (1)(4)(23), (4)(123), (4)(132).$$

The new polynomial will simply have another f_1 in every term, so it can be factored out, and the new polynomial will simply have an additional factor of $(x_1 + x_2 + x_3 + \cdots + x_n)$ (assuming you are analyzing the situation with n colors. The total count will thus simply be n times the previous count, as we noted above.

To make this concrete, look at this triangle case with three colors. Ignoring slot 4, we have:

$$P = \frac{f_1^3 + 3f_1f_2 + 2f_3}{6}.$$

If we include slot 4, we get:

$$P = \frac{f_1^4 + 3f_1^2f_2 + 2f_1f_3}{6} = \frac{f_1(f_1^3 + 3f_1f_2 + 2f_3)}{6}.$$

If we substitute $(x + y + z)$ in the usual way, we obtain:

$$P = x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + xyz,$$

and for P' we obtain:

$$\begin{aligned} P' &= (x + y + z)P \\ &= (x + y + z)(x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + xyz). \end{aligned}$$

The result will be all the previous terms with an extra x term, all the previous terms with an additional y term, and similarly for z. That means for every valid coloring in the simpler situation, there will be one more with each of the available colors, as we expect.

3.5 Independent Parts

Assume that the allowable symmetries are, in a sense, disconnected, meaning that we can divide the slots into two groups and all the available symmetries move the slots within the groups, but there are none that move the contents of a slot from one group to another.

As a simple example, imagine a child's rattle toy that has hollow balls on both ends of a handle, and one of the hollow balls contains 3 marbles while the other contains 2. In how many ways can this rattle be filled with marbles of 4 different colors? Or perhaps a more practical (and almost equivalent) example is from chemistry: Imagine a carbon atom hooked to a nitrogen atom via a single bond. You can connect three other atoms to the carbon and two other atoms to the nitrogen. If the other atoms to be hooked on are chosen from among hydrogen, fluorine, chlorine, and iodine, how many different types of chemicals are possible[1]?

In the rattle example, there is no ordering to the three marbles on one end of the rattle and to the two on the other end, but the two ends cannot be swapped since they contain different numbers of marbles. All the symmetries involve swapping among the three or among the two. So if 1, 2,

[1] Although it may seem that these two examples are identical, they are not—the marbles in the three side can be swapped in any way (so there are 6 symmetries); on the carbon atom, they can only be rotated (so there are only 3 symmetries of the carbon-attached atoms).

and 3 represent the slots for marbles on the three side, and if 4 and 5 represent those on the two side, we can take what's called mathematically a "direct product" of the individual groups to get the symmetry group for the entire rattle.

All the entries in the table below form the symmetry group for the rattle as a whole. Each is composed of a product of one symmetry from the three side and one from the two side:

	(1)(2)(3)	(1)(23)	(3)(12)	(2)(13)	(123)	(132)
(4)(5)	(1)(2)(3)(4)(5)	(1)(4)(5)(23)	(3)(4)(5)(12)	(2)(4)(5)(13)	(4)(5)(123)	(4)(5)(132)
(45)	(1)(2)(3)(45)	(1)(23)(45)	(3)(12)(45)	(2)(13)(45)	(45)(123)	(45)(132)

Thus, when we have a term like $f_1 f_2$ from the three group and an element like f_1^2 in the two group, the combination will simply generate a term that is the product of the two: $f_1^3 f_2$, and this will happen in every case. It should be easy to see (if you don't see it, work out the polynomials and check) that if P_2 is the cycle index polynomial for the two group and P_3 is the one for the three group, then the cycle index polynomial for the entire permutation group will simply be $P_2 P_3$, and it's clear that the counts of possible configurations will simply be the products of the individual configurations.

3.6 Cyclic Permutations

Next, let's look at one example that is still simple, but a bit more complicated than what we've examined up to now—we'll examine the case where the positions can be rotated by any amount, but cannot be flipped over. For concreteness, assume that you've got a circular table, and you wish to set the table with plates of k different colors, but rotations of the plates around the table are considered to be equivalent. In how many different ways can this be done?

It seems that cyclic permutations are pretty simple, but as you'll see, at least a little care must be taken. We'll examine two examples that seem similar at first, but illustrate most of the interesting behavior that you can see. We'll look at the groups of cyclic permutations of both 6 and 7 elements.

Call the positions (or slots) 1, 2, 3, 4, 5, and 6 (for the table with 6 place settings), and we'll add position 7 for the table with seven. Listed below are the complete sets of cyclic permutations of 6 or 7 objects.

	1	2	3	4	5	6
(1)(2)(3)(4)(5)(6)	1	2	3	4	5	6
(123456)	2	3	4	5	6	1
(135)(246)	3	4	5	6	1	2
(14)(25)(36)	4	5	6	1	2	3
(153)(264)	5	6	1	2	3	4
(165432)	6	1	2	3	4	5

	1	2	3	4	5	6	7
(1)(2)(3)(4)(5)(6)(7)	1	2	3	4	5	6	7
(1234567)	2	3	4	5	6	7	1
(1357246)	3	4	5	6	7	1	2
(1473625)	4	5	6	7	1	2	3
(1526374)	5	6	7	1	2	3	4
(1642753)	6	7	1	2	3	4	5
(1765432)	7	1	2	3	4	5	6

Notice that the lower table (for 7 elements) every permutation except for the identity has the same form (in terms of cycle structure), while the table with 6 elements has a variety of cycle structures. The reason, of course, is that 7 is a prime number. With the 6-element example, three rotations of two positions or two rotations of three positions bring you back to where you started. If the plates on the 6-table are colored "red, green, red, green, red, green", they rotate to themselves after every rotation of 2 positions, or if the coloring is "red, green, blue, red, green, blue" they rotate to themselves after a rotation of three positions. In fact, it's easy to see that something similar will happen for any integer multiples of the table size. If the table, however, has a prime number of positions, the only way to bring it back to the initial configuration is to leave it alone, or turn it through an entire 360° rotation.

Thus if we are counting colorings that are unique when taking rotations into account, we should expect different behavior if the number of positions is prime or not. Clearly the cycle indices of the two examples above look quite different:

$$P_6 = \frac{f_1^6 + f_2^3 + 2f_3^2 + 2f_6}{6},$$

and

$$P_7 = \frac{f_1^7 + 6f_7}{7}.$$

There is a lot more detail about cyclic (and dihedral) groups in Appendix C.4.

4 Proof that Pólya's Method Works

To prove that Pólya's counting method works we will need to prove a few results in group theory. We will only need to work with groups of permutations, and an introduction to permutations is covered in Appendix A.

We have used Pólya's method to perform two kinds of counts, so there are two things to prove. Both count colorings of objects having symmetries that consider some colorings to be equivalent to others. Given the object and its symmetries, one application of the method tells us how many different colorings there are if k different colors can be used but there is no constraint on the number of slots on the object that are colored with any particular color. The other application of the method is similar, but it allows us to choose a fixed number of slots to assign to each color, as in, "How many ways can the object be colored with three red, four blue, and seven yellows?"

In addition to the properties of permutations, you will need to know some simple results in group theory including the definition of a group and a subgroup, and how to work with cosets based on a group and its subgroup. If you are unfamiliar with these topics or if you wish to review them, they are covered in Appendix B.

4.1 Groups Acting on Sets

We are interested here in how a group G of permutations acts on a set X. As a concrete example consider first the striped cloth problem that we examined in Section 2.1. For concreteness again, let's consider a piece of cloth with 3 stripes colored with three different colors: red (R), yellow (Y), and blue (B). The group G consists of two permutations: one (the identity, or $(1)(2)(3)$) leaves the stripes as they are and the other, $(2)(13)$, reverses them.

If we look at pieces of cloth and don't have the option of reversing them, there are $3^3 = 27$ colorings, listed, say, from left to right.

RRR	RRY	RRB	RYR	RYY	RYB	RBR	RBY	RBB
YRR	YRY	YRB	YYR	YYY	YYB	YBR	YBY	YBB
BRR	BRY	BRB	BYR	BYY	BYB	BBR	BBY	BBB

These 27 colorings compose the set X and the two elements of G operate on them. For example, the YRB coloring has a yellow in slot 1, a red in slot 2, and a blue in slot 3. The identity operates on an element of X by leaving it unchanged; the reversing permutation swaps the first and last colors (the colors in slots 1 and 3), so it will map $RRB \to BRR$, $RYB \to BYR$, $RRR \to RRR$, $YBY \to YBY$, et cetera. Notice how the first permutation fixes all elements of X and the second fixes only some of them (it fixes the ones whose first and last colors are the same). A group element will map each member of X to a member of X, which may be the same or different, but generally, each element of G maps every member of X to some member of X.

In this article, all the examples will look like this: elements of the permutation group G will move the things (often colors, or atoms) among the slots of the larger object that contains all the slots. The set X will list every possible arrangement of colors in those slots without worrying about whether they are to be treated as the same as others, given the permutations. Sometimes it is useful to think of the members of X as functions assigning a color to each slot.

If $g \in G$ and g acts on $x \in X$ to produce $y \in X$, we write $g(x) = y$. If g is the reversing permutation in the example above, $g(RRB) = BRR$, et cetera.

4.2 Orbits

Notice that we can divide the elements of the set X into "orbits" where each element in an orbit can be mapped to any other element in the orbit by some permutation in the group G. In the case of the three-striped cloth the orbits are of size at most 2, since there are only 2 members of G. Here are some of the orbits for this example:

$$\{RRR\}, \{RRY, YRR\}, \{RRB, BRR\}, \{RYR\}, \{RYY, YYR\}, \ldots$$

In this case, the orbit has size 1 if every element of G maps it to itself (in other words, if it is symmetric), and the orbit has size 2 if it looks different after turning it around. For a more complicated group or set X, there may be lots of orbits of many different sizes. The members of X that are filled entirely with the same color always form orbits of size 1, since things like YYY have to go to YYY since no matter how you swap the Y's around, the result will look the same.

Basically, the orbit of a member x of the set X (notation: $Orb(x)$) is the set of all members that elements of the group G can map it to:

$$Orb(x) = \{g(x) : g \in G.\}$$

If one element a is mapped to another element b in the orbit by $g \in G$, then b is mapped to a by $g^{-1} \in G$. The inverse, g^{-1}, is guaranteed to be in G since G is a group.

In the particular example above, there are 18 orbits in X under G. The members of each orbit all represent colorings that we are supposed to consider to be "the same." Thus counting the number of orbits solves the sort of problem that can be handled by Pólya's method where we consider every possible way to assign colors to the object.

If we ask, "How many different molecules are possible if we hook hydrogen (H), fluorine (F), and iodine (I) atoms to the six positions of a benzene ring?" this is equivalent to, "How many orbits are there for the dihedral group[2] on six elements acting on the set of all possible 6-tuples of H, F, and I?"

There are $3^6 = 729$ colorings, but the symmetries of benzene divide them into a much smaller set of orbits. Here are three orbits of this set of colorings where the atoms are listed in clockwise order from a fixed position on the benzene ring:

$$\begin{aligned}
&\{HHHHHH\}\\
&\{IHIHHH, HIHIHH, HHIHIH, HHHIHI, IHHHIH, HIHHHI\}\\
&\{IFHHHH, HIFHHH, HHIFHH, HHHIFH, HHHHIF, FHHHHI,\\
&\quad FIHHHH, HFIHHH, HHFIHH, HHHFIH, HHHHFI, IHHHHF\}
\end{aligned}$$

The other question we will solve is how to count the number of orbits that have the same numbers of items to fill the slots. Using the chemistry benzene example again, there are three different orbits that have exactly three hydrogen and three iodine atoms attached to the six positions on the benzene ring. In this case here are the three different orbits:

$$\begin{aligned}
&\{HHHIII, HHIIIH, HIIIHH, IIIHHH, IIHHHI, IHHHII\}\\
&\{HHIHII, HIHIIH, IHIIHH, HIIHHI, IIHHIH, IHHIHI,\\
&\quad IIHIHH, HIIHIH, HHIIHI, IHHIIH, HIHHII, IHIHHI\}\\
&\{HIHIHI, IHIHIH\}
\end{aligned}$$

Every item in each orbit represents exactly the same chemical, meaning that there are exactly three different chemicals that can be made by hooking three hydrogen atoms and three iodine atoms to a benzene ring.

[2]The "dihedral group" on n elements is the mathematical name for the group of all rotations and mirror reflections of a ring of n slots.

4.3 Stabilizers

For a given $x \in X$, some elements $g \in G$ fix x: $g(x) = x$, and some act on x to produce something different from x. For any particular x, an interesting subset of G (which we will prove to be a subgroup of G) is called the "stabilizer of x" (notation: $Stab(x)$):

$$Stab(x) = \{g : g \in G \text{ and } g(x) = x\}.$$

We now show that $Stab(x)$ is a subgroup of G.

- Identity: The identity e is in $Stab(x)$ since $e(x) = x$.
- Closure: If g_1 and g_2 are in $Stab(x)$, then $g_1(g_2(x)) = g_1(x) = x$, so $g_1g_2 \in Stab(x)$.
- Inverses: If $g \in Stab(x)$ we need to show that $g^{-1} \in Stab(x)$ meaning that $g^{-1}(x) = x$. Since $g \in Stab(x)$ we know that $g(x) = x$. $e = g^{-1}g$ so $x = e(x) = g^{-1}(g(x)) = g^{-1}(x)$, so $x = g^{-1}(x)$, so $g^{-1} \in Stab(x)$.

Since $Stab(x)$ is a subgroup of G, we can consider the cosets of $Stab(x)$. If $g \in G$, then look at the coset $g\ Stab(x)$. For any $h \in Stab(x)$, $g(h(x)) = g(x)$, so every element in the coset maps x to the same place.

If g_1 and g_2 both map x to the same value then g_1 and g_2 are in the same coset. To show this, suppose $g_1(x) = g_2(x)$. Then $g_2^{-1}g_1(x) = x$, so $g_2^{-1}g_1 \in Stab(x)$. Thus for some $h \in Stab(x)$ $g_2^{-1}g_1 = h$ so $g_1 = g_2h$. So the coset $g_1\ Stab(x) = g_2h\ Stab(x)$ but since $h \in Stab(x)$ we have $g_2h\ Stab(x) = g_2\ Stab(x)$, and so $g_1\ Stab(x) = g_2\ Stab(x)$.

Since the cosets are disjoint, are all the same size, and together include all values of $g \in G$, every coset corresponds to a different element in the orbit of x. Thus we have:

$$|G| = |Stab(x)| \cdot |Orb(x)|.$$

(When we place the "absolute value bars" around a set or group, it means "the number of objects in that set or group." So $|G|$ is the number of objects in G.)

Since the formula above is true for every $x \in X$, we have, for every $x_i \in Orb(x)$:

$$|G| = |Stab(x_i)| \cdot |Orb(x_i)| = |Stab(x_i)| \cdot |Orb(x)|.$$

In other words, the stabilizers of each of the elements in the same orbit are the same size.

4.4 Statement and Proof of Burnside's Theorem

Burnside's theorem gives the relationship between the number of orbits and the number of elements fixed by particular permutations in the group G. Here is a statement of the theorem:

Theorem 1 (Burnside) *If X is a finite set and G is a group of permutations that act on X, let $Fix(g)$ be the number of elements of X that are fixed by a particular $g \in G$. Then the number N*

of different orbits of X under G is given by:

$$N = \frac{1}{|G|}\sum_{g\in G} Fix(g).$$

In other words, the number of orbits is the average size of the sets of items fixed by elements of G.

To see that it works for the example above with the three-striped reversible cloth and 3 colors, $F(e) = 27$ (e is the identity of G), and $F(g) = 9$, where g is the permutation that reverses the three colors. $F(g) = 9$ because the center color can be any of three, but once we pick one of three colors for one end of the strip, it must have the same color at the other end, so there are $3 \cdot 3 = 9$ fixed colorings. We have $|G| = 2$, so the number of orbits is $(1/2)(9 + 27) = 18$, which is what we determined previously.

Proof of Burnside's Theorem:

Consider the set S of pairs consisting of a group element g and a member x of X such that x is fixed by g. Mathematically, $S = \{(g, x) : g \in G \text{ and } g \text{ fixes } x\}$. Then:

$$|S| = \sum_{g\in G} Fix(g),$$

since $Fix(g)$ is the number of members in X that g fixes.

But another way to express the size of S is:

$$|S| = \sum_{x\in X} |Stab(x)|,$$

since the stabilizer of x contains all the elements in G that fix x.

Now, suppose that there are N orbits. Select N elements: $x_1, x_2, \ldots, x_N$ such that each one of them is in a different orbit. We then have:

$$|S| = \sum_{x\in X} |Stab(x)| = \sum_{i=1}^{N} \sum_{x\in Orb(x_i)} |Stab(x)| = \sum_{i=1}^{N} |Orb(x_i)||Stab(x_i)| = N \cdot |G|.$$

From this we have:

$$\sum_{g\in G} Fix(g) = |S| = \sum_{x\in X} |Stab(x)| = N \cdot |G|.$$

Divide though by $|G|$ to obtain Burnside's theorem:

$$N = \frac{1}{|G|}\sum_{g\in G} Fix(g).$$

Another way to look at this is that N, the number of orbits, is the average of the number of elements in X that are fixed by members of G.

4.5 Proof of Pólya's Counting Method

We now have enough machinery to prove the validity of the counting method to obtain the total number of orbits.

Recall that Pólya's method counts distinct colorings of various objects where certain colorings are considered to be equivalent if any of the given symmetry operations map them to each other. In the earlier part of this section, the set X of items to be operated upon represented the set of all colorings where each was considered different.

For any element $g \in G$ we would like to calculate $Fix(g)$ so we can use Burnside's theorem. That is actually easy to do if we know the cycle structure of the group element g. Let's look at a g that is far more complex than in any of the examples so far and think about how we might be able to calculate $Fix(g)$ for a situation with k colors.

Suppose g looks like this:

$$g = (1)(2)(3\ 4)(5\ 6)(7\ 8)(9\ 10\ 11\ 12\ 13\ 14).$$

What sorts of colorings will g fix? It will always map slot 1 to slot 1 and slot 2 to slot 2 so it doesn't matter what colors those have. Since it swaps slots 3 and 4 (as well as 5 and 6 and also 7 and 8), then as long as both items in those pairs of slots are the same, g will fix them. Finally, all the slots from 9 to 14 must be the same or g will change the color of at least one slot and so will not fix that coloring.

So if we choose 6 colors (not necessarily different) to assign to each of the cycles where every element in each cycle is colored the same, then g will fix that coloring. If there are k different colors, then there are k^6 colorings that will be fixed by g.

In other words, a coloring fixed by g has to look something like this (where the colors are listed in the order of the slots):

$$C_1C_2C_3C_3C_4C_4C_5C_5C_6C_6C_6C_6C_6C_6,$$

where each C_i can be any of the k colors.

Let's let $k = 3$ for illustration purposes. The term corresponding to g in the numerator of the cycle-index polynomial will look like this:

$$(x+y+z)(x+y+z)(x^2+y^2+z^2)(x^2+y^2+z^2)(x^2+y^2+z^2)(x^6+y^6+z^6). \qquad (5)$$

There are basically 6 trinomials in the product and if you multiply out the whole thing without combining like terms you will have 3^6 terms where you have selected one item from the first group, one from the second, one from the third, and so on, up to one item in the sixth term.

However the choices are made, each final term will have exponents that sum to 14 with two first powers from the first two terms, three second powers from the next three terms, and one sixth power from the final term. If we substitute $x = y = z = 1$ in the term each of the groups in parentheses will evaluate to 3, so there will 3^6 total terms, each corresponding to a pattern of colors that is fixed, so this is a way to calculate

$$N = \frac{1}{|G|}\sum_{g \in G} Fix(g).$$

Obviously if there are k colors available, each factor in parentheses would have sums of k variables and if all k variables were set to 1 there would be k^6 fixed colorings fixed by g.

There is nothing special about the g that we used, and every element of G will have a cycle structure that can be interpreted in a similar way, showing that Bernstein's theorem gives us a way to count the total number of colorings (equivalently, orbits) if we can find the cycle index of G.

In other words, $Fix(g)$ for this particular g will be k^6, and there will be k^6 terms generated by multiplying out the terms in Equation 5. For the terms from that group element g, if you set all k variables to 1 the terms will add to k^6 which is the number of colorings fixed by g.

4.6 Weighting

To state Pólya's method precisely and to prove that it works, it is best to give precise definitions of the things we have been working with up to now.

Suppose that the object we are working with has d different slots in the set

$$D = \{s_1, s_2, \dots, s_d\},$$

and if there are k colors, then the set of all the colors that can be used is:

$$C = \{c_1, c_2, \dots, c_k\}.$$

Then the set X of all possible colorings is the set of all mappings from D to C. We can write this formally[3] as follows:

$$X = \{x | x : D \to C\}.$$

The size of X is k^d ($|X| = k^d$).

We will assign to each color $c_i \in C$ a weight $w(c_i)$. Think of that weight as being a positive real number that we can either state specifically or simply use a variable (like the x, y, and z we've been using to indicate the presence of a color). To get the weight of a particular coloring we need to multiply all the weights of the colors in that coloring. So if $x \in X$ we define the weight of x (or $w(x)$) as:

$$w(x) = \prod_{d_i \in D} w(x(d_i)).$$

That is, you multiply together the weights of the colors in all the slots.

The definition above has a very nice feature: if $g \in G$ maps one coloring to another, then both of those colorings will have the same weight since g just rearranges the colors in the slots. That means that every member of x's orbit will have the same weight as x.

4.7 The Pattern Inventory

The group G will divide X into a number of orbits, say, O_1, O_2, $\dots O_N$. Every member of an orbit will have the same weight, so there is no ambiguity in defining the weight of an orbit to be

[3] Mathematicians often write this as C^D since if C and D were just numbers then the exponential form would give the number of different mappings. (There are C possibilities for each value of D so there are $|C|^{|D|}$ such mappings.)

the weight of any element in it. We will define the pattern inventory to be:

$$PI = w(O_1) + w(O_2) + \cdots + w(O_N).$$

4.8 The Cycle Index Polynomial

Every $g \in G$ has a certain cycle structure. Suppose there are k_1 cycles of length 1, k_2 cycles of length 2. et cetera. The longest possible cycle would have length $d = |D|$. The cycle index for g will be:

$$ci(g) = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}.$$

(Most of the k_i will usually be zero.)

The cycle index polynomial is simply the sum of all the cycle indices:

$$C_G(x_1, x_2, \ldots x_d) = \sum_{g \in G} ci(g).$$

4.9 Pólya's Inventory Theorem

Theorem 2 (Pólya's Inventory Theorem) *Let* $S_j = \sum_{i=1}^{k} (w(c_i))^j$. *Then:*

$$PI = C_G(S_1, S_2, \ldots, S_d).$$

Proof: Divide X into a set of equivalence classes X_1, X_2, …, X_m, such that $x \sim y$ if $w(x) = w(y)$. Note that each X_i is the union of some set of orbits. Every element of an orbit has the same weight, but two different orbits may also contain elements having the same weight. Every orbit will be contained in exactly one of the X_i.

Since every $g \in G$ maps elements of the same weight to elements of the same weight, every g maps elements of X_i into other elements of X_i, for every i. Let $g^{(i)}$ be the result of restricting g to X_i.

Let m_i be the number of orbits of these restricted $g^{(i)}$ in the set X_i. Then we can write PI as follows:

$$PI = \sum_{i=1}^{m} m_i W_i,$$

where W_i is the common weight of every coloring in X_i.

But:

$$m_i = \frac{1}{G} \sum_{g \in G} |Fix(g^{(i)})|$$

by Burnside's theorem, so:

$$\begin{aligned} PI &= \sum_{i=1}^{m} W_i \Big(\frac{1}{G} \sum_{g \in G} |Fix(g^{(i)})| \Big). \\ &= \frac{1}{G} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g^{(i)})| W_i \\ &= \frac{1}{G} \sum_{g \in G} W(Fix(g)) \end{aligned}$$

Since $Fix(g)$ is equal to the union of all the $Fix(g^{(i)})$.

Finally, if

$$ci(g) = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}.$$

We have:

$$W(Fix(g)) = \Big(\sum_{i=1}^{k} (W(c_i))^1 \Big)^{k_1} \Big(\sum_{i=1}^{k} (W(c_i))^2 \Big)^{k_2} \cdots \Big(\sum_{i=1}^{k} (W(c_i))^d \Big)^{k_d}.$$

Substituting, we obtain:

$$\begin{aligned} PI &= \frac{1}{G} \sum_{g \in G} \Big(\sum_{i=1}^{k} (W(c_i))^1 \Big)^{k_1} \Big(\sum_{i=1}^{k} (W(c_i))^2 \Big)^{k_2} \cdots \Big(\sum_{i=1}^{k} (W(c_i))^d \Big)^{k_d}. \\ &= C_G(S_1, S_2, \ldots, S_d). \end{aligned}$$

4.10 Other Applications of Pólya's Inventory Theorem

Once we have calculated the cycle index polynomial in k variables (assuming there are k possible colors) we can set all the variables to 1 and this will give us the total number of possible colorings with any arrangement of the colors. In fact, if this is what is desired, it is easiest to substitute 1 for the variables before the cycle index polynomial is expanded.

If we are looking for the number of possible colorings with specific numbers of each of the k colors to be used, we can expand the cycle index polynomial and look only for terms with the correct weights. The sum of those coefficients will solve our problem. In many cases we do not even need to do a complete expansion of the cycle index polynomial since we can use multinomial coefficients to find the specific values of the coefficients for terms that interest us.

There are other sorts of questions to which the inventory theorem can be applied. Suppose there are k colors to be put into d slots, and we are interested only in counting the number of colorings that include exactly 2 red slots. If x is the weight of red, we can expand the cycle index polynomial leaving x as it is, but substituting 1 for all the other $d-1$ variables. Then look for terms that have x^2 terms and add those coefficients to obtain the result. Obviously this idea can be extended to situations where you have more than one of the colors locked down, but you don't care about the numbers of other colors that are used.

Finally, again just by counting terms and combining the coefficients, you can answer questions like, "How many colorings have at least two reds?" Or, "How many colorings have at least two reds and fewer than four yellows?" For this last question, assuming that you use x for red's weight and y for yellow's weight, you leave x and y as-is in the cycle index polynomial, set all the other variables to 1, expand it, and look for terms where the exponent of x is 2 or more and at the same time, the coefficient of y is three or less.

5 A Non-Trivial Practical Example

In this section we will consider the problem of counting the number of isomers of molecules that are derivatives of the chemical ethane (see Figure 2). Ethane is composed of two connected carbon atoms (marked with a "C" in the figure) and each of those carbon atoms is (roughly) the center of a tetrahedral arrangement of bonds. One of those bonds goes to the other carbon atom, and the other three go to other chemical groups. In the figure, positions 1, 2 and 3 are hooked to one carbon and 4, 5, and 6 are hooked to the other.

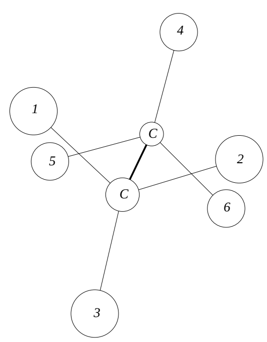


Figure 2: Ethane Compounds

Pure ethane has a hydrogen atom at all six positions, but any or all of the positions can be replaced by other chemical groups. The molecule can rotate freely about the carbon-carbon bond. Thus if the groups at positions 1, 2 and 3 are rotated by the permutation $(1\ 2\ 3)$, the molecule is equivalent. But the direction cannot be reversed without changing the molecule: $(1)(2\ 3)$ makes a different molecule.

If you look along the carbon-carbon bond from the end with positions 1, 2 and 3 on it, the 1, 2, 3 positions appear in clockwise order, and if you look toward the other end, the 4, 5, and 6 also appear in clockwise order, as in the figure.

So the symmetry operations that leave the molecule unchanged involve any rotation about either carbon atom, or turning the molecule around (basically exchanging positions 1 and 4, 2 and 5, and 3 and 6).

The group of permutations consists of 18 symmetries: three rotations on each end (for $3 \times 3 = 9$ of them), and then flipping the molecule's carbon atoms doubles that. The following table shows all of those symmetry permutations written in terms of the three primitive rotations (which we shall call a and b) and the carbon-carbon flip (which we shall call c). (In fact, it can be generated by just a and c or just by b and c, but the expressions (but not the permutations themselves) become more complicated.)

$e = (1)(2)(3)(4)(5)(6)$	f_1^6	$c = (1\,4)(2\,5)(3\,6)$	f_2^3
$a = (4)(5)(6)(1\,2\,3)$	$f_1^3 f_3^1$	$ac = (1\,5\,2\,6\,3\,4)$	f_6^1
$a^2 = (4)(5)(6)(1\,3\,2)$	$f_1^3 f_3^1$	$a^2c = (1\,6\,3\,5\,2\,4)$	f_6^1
$b = (1)(2)(3)(4\,5\,6)$	$f_1^3 f_3^1$	$bc = (1\,4\,2\,5\,3\,6)$	f_6^1
$b^2 = (1)(2)(3)(4\,6\,5)$	$f_1^3 f_3^1$	$b^2c = (1\,4\,3\,6\,2\,5)$	f_6^1
$ab = (1\,2\,3)(4\,5\,6)$	f_3^2	$abc = (1\,5\,3\,4\,2\,6)$	f_6^1
$ab^2 = (1\,2\,3)(4\,6\,5)$	f_3^2	$ab^2c = (1\,5)(2\,6)(3\,4)$	f_2^3
$a^2b = (1\,3\,2)(4\,5\,6)$	f_3^2	$a^2bc = (1\,6)(2\,4)(3\,5)$	f_2^3
$a^2b^2 = (1\,3\,2)(4\,6\,5)$	f_3^2	$a^2b^2c = (1\,6\,2\,4\,3\,5)$	f_6^1

When we combine the 18 pattern-inventory polynomials from the table above, we obtain:

$$P = \frac{f_1^6 + 4f_1^3 f_3^1 + 4f_3^2 + 3f_2^3 + 6f_6^1}{18}. \tag{6}$$

The exponents of 1 are obviously unnecessary, but they are included to emphasize the fact that there is exactly one cycle of that length in the permutation.

If we are using k different kinds of chemical groups to be hooked to the 6 positions in the ethane-derivative molecules, then:

$$f_i = (c_1^i + c_2^i + \cdots + c_k^i),$$

where the c_j corresponds to the j^{th} chemical group.

Let's look at the situation with different numbers of groups (colors), beginning with the simplest situation: one group. Obviously, if all six positions are filled with the same group, there is only one way to do it. In Equation 6, if there is one group, $f_i = c_1^i$, yielding:

$$P = \frac{c_1^6 + 4c_1^6 + 4c_1^6 + 3c_1^6 + 6c_1^6}{18} = \frac{18c_1^6}{18} = c_1^6.$$

The coefficient 1 of c_1^6 means that there is exactly one way to construct an ethane derivative if all six attached groups are the same.

Things get a bit more interesting with two different kinds of groups:

$$\begin{aligned}
P &= \frac{(c_1+c_2)^6 + 4(c_1+c_2)^3(c_1^3+c_2^3) + 4(c_1^3+c_2^3)^2 + 3(c_1^2+c_2^2)^3 + 6(c_1^6+c_2^6)}{18} \\
&= c_1^6 + c_1^5c_2 + 2c_1^4c_2^2 + 2c_1^3c_2^3 + 2c_1^2c_2^4 + c_1c_2^5 + c_2^6 \\
&= c_1^6 + c_2^6 + \\
&\quad c_1^5c_2 + c_1c_2^5 + \\
&\quad 2c_1^4c_2^2 + 2c_1^2c_2^4 + \\
&\quad 2c_1^3c_2^3.
\end{aligned}$$

Here is the expansion with three groups:

$$
\begin{aligned}
P &= \frac{\begin{array}{c}(c_1+c_2+c_3)^6+4(c_1+c_2+c_3)^3(c_1^3+c_2^3+c_3^3)+\\ 4(c_1^3+c_2^3+c_3^3)^2+3(c_1^2+c_2^2+c_3^2)^3+6(c_1^6+c_2^6+c_3^6)\end{array}}{18}\\
&= c_1^6+c_1^5c_2+c_1^5c_3+2c_1^4c_2^2+3c_1^4c_2c_3+2c_1^4c_3^2+\\
&\quad 2c_1^3c_2^3+4c_1^3c_2^2c_3+4c_1^3c_2c_3^2+2c_1^3c_3^3+\\
&\quad 2c_1^2c_2^4+4c_1^2c_2^3c_3+6c_1^2c_2^2c_3^2+4c_1^2c_2c_3^3+2c_1^2c_3^4+\\
&\quad c_1c_2^5+3c_1c_2^4c_3+4c_1c_2^3c_3^2+4c_1c_2^2c_3^3+3c_1c_2c_3^4+c_1c_3^5+\\
&\quad c_2^6+c_2^5c_3+2c_2^4c_3^2+2c_2^3c_3^3+2c_2^2c_3^4+c_2c_3^5+c_3^6\\
&= c_1^6+c_2^6+c_3^6+\\
&\quad c_1^5c_2+c_1^5c_3+c_1c_2^5+c_1c_3^5+c_2^5c_3+c_2c_3^5+\\
&\quad 2c_1^4c_2^2+2c_1^4c_3^2+2c_1^2c_2^4+2c_1^2c_3^4+2c_2^4c_3^2+2c_2^2c_3^4+\\
&\quad 4c_1^3c_2^2c_3+4c_1^3c_2c_3^2+4c_1^2c_2^3c_3+4c_1^2c_2c_3^3+4c_1c_2^3c_3^2+4c_1c_2^2c_3^3+\\
&\quad 3c_1^4c_2c_3+3c_1c_2^4c_3+3c_1c_2c_3^4+\\
&\quad 2c_1^3c_2^3+2c_1^3c_3^3+2c_2^3c_3^3+\\
&\quad 6c_1^2c_2^2c_3^2
\end{aligned}
$$

In each of the final two cases above, the product is rearranged to show that the coefficients on "similar" expressions is the same. For example, in the expansion with three groups, every term that consists of four copies of one color and two of another has the same coefficient of 2: $2c_1^4c_2^2$, $2c_1^4c_3^2$, $2c_1^2c_2^4$, et cetera. This makes perfect sense, of course, since there should be the same number of ways to construct an ethane derivative with four hydrogens and two chlorines as to construct one with four chlorines and two bromines.

Another thing to notice is that each expansion includes all the previous ones. If you set $c_3 = 0$ in the third one, you obtain exactly the second one, et cetera.

The examples above provide plenty of examples to check that the counting scheme works, and as exercises in counting for you. For example, with the three different groups, apparently there are six fundamentally different molecules with two of each type. Can we find them all? If the three groups are called a, b and c, how can they be divided into two groups of three to be added to each carbon atom? Here's a complete list:

$$
\begin{array}{c}
aab \leftrightarrow bcc\\
aac \leftrightarrow bbc\\
abb \leftrightarrow acc\\
abc \leftrightarrow abc
\end{array}
$$

Why are there only four? The reason is that although the first three examples have two of one and one of another on both sides, the $abc \leftrightarrow abc$ has three different types on each carbon atom, and those can be arranged clockwise or counter-clockwise as we look at the molecule along the carbon-carbon bond. So that fourth division allows three different molecule types: both arranged clockwise, both counter-clockwise, and one arranged in each direction, for a total of six, as predicted by Pólya's counting method.

A Permutations

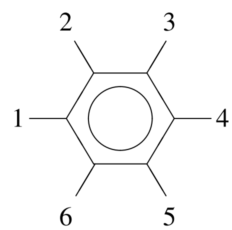
This appendix provides a (very gentle) introduction to permutations. There are various ways to think about them, and the approach taken here provides a useful way to think about permutations that works well for the examples in this article.

Basically, a permutation is a rearrangement of objects. We would like to make it possible to consider every possible way to rearrange things including even the trivial: "leave everything exactly where it was."

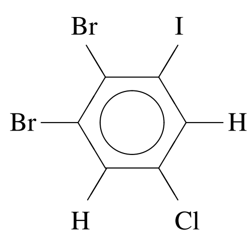
A.1 Slots and Objects

As a concrete example let's use the benzene molecule from the first section of this article where the objects to be rearranged are the atoms hooked to the six carbons that form the benzene ring. We can think of the ring as having six "slots," each of which will hold an atom like hydrogen, bromine, chlorine, iodine, et cetera.

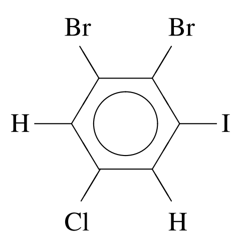
For a fixed orientation of the central carbon ring, let's number the slots to be filled as follows:



and let's consider a particular assignment of atoms to those slots:



To draw the same molecule above but where the drawing is rotated by one position clockwise, we would obtain this:



The rotation above of the molecule could have been described as follows:

"Move the object in slot 1 to slot 2, move the object in slot 2 to slot 3, move the object in slot 3 to slot 4, and so on, until finally, move the object that was in slot 6 into slot 1."

A more mathematical way to describe this rotation is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}.$$

where the numbers in the top line indicate the slot from which an object (an atom in this case) is moved and the number below each is the slot to which that object is moved.

A.2 Cycle Notation

A more compact and useful way to represent a permutation is the so-called "cycle notation:"

$$(1\,2\,3\,4\,5\,6).$$

To see where the contents of a slot go, look to the right of that slot's number, or, if the cycle ends, move it to the first slot mentioned in the cycle. In this simple permutation where the positions basically move in a circle, there is one long cycle of length 6.

More interesting things happen if we move each object two positions or three positions clockwise. If we move two positions, we get the following cycle representation:

$$(1\,3\,5)(2\,4\,6).$$

Make sure you understand this. Here is the cycle representation for a permutation that shifts each object three positions clockwise:

$$(1\,4)(2\,5)(3\,6).$$

In other words, the objects in slots 1 and 4 are swapped, as are those in slots 2 and 5, and those in slots 3 and 6.

Some of the permutations leave some objects in the same slot. For example, imagine turning the molecule over along the 1-4 axis. This will leave the objects in slots 1 and 4 where they were, and will swap slots 2 and 6 as well as 3 and 5. Here's the cycle representation for that operation:

$$(1)(2\,6)(3\,5)(4).$$

Often, in other articles and books, the "cycles" of length 1 are omitted, but here we will always list them. The disadvantage is that the representation may be a bit longer but an advantage is that you can check to make sure that every slot's movement is listed. There are other advantages, too.

A.3 Equivalent Representations of Permutations

Since the cycles in all of the examples above are disjoint the order of the cycles doesn't matter, so here are a few equivalent ways to display this last example:

$$(1)(2\,6)(3\,5)(4) = (1)(4)(2\,6)(3\,5) = (2\,6)(1)(4)(3\,5) = (3\,5)(2\,6)(1)(4).$$

Also keep in mind that a single cycle with more than one slot can be presented in different ways. For example:

$$(1\,2\,3) = (2\,3\,1) = (3\,1\,2).$$

A.4 A Canonical Representation of Permutations

Mathematicians use the word "canonical" to mean a representation that is standard and that makes it easy to compare items that might look different. In elementary school you probably learned that the canonical way to represent a fraction was to reduce it to lowest terms (but it's unlikely that your teacher used the word "canonical"). Although $1/2 = 2/4 = 3/6 = 4/8 = \ldots$ we usually reduce them to lowest terms so that it is easy to see that $2/4 = 17/34$. Reduce both to $1/2$, and so they must be equivalent.

There are different canonical ways to write a permutation in this cycle form which make it easy to compare two of them to see if they are the same. Here is what works best for the examples in this article:

1. List all cycles of length 1 (if there are any). Then list all cycles of length 2 (if there are any), then those of length 3, et cetera.
2. Since the cycles can start at any of the slots, begin with the slot that has the smallest-number (or, if the slots are named with letters, the slot with the earliest position in the alphabet).
3. If there are multiple cycles of the same length, the first one listed will be the one with the smallest number in the first position (or the earliest letter in the alphabet).

Here is an example of a (complex) permutation on the left and its canonical form on the right:

$$(11\ 4\ 5)(7\ 12)(9)(8\ 6\ 2\ 1\ 3)(13\ 10)(14) = (9)(14)(7\ 12)(10\ 13)(4\ 5\ 11)(1\ 3\ 8\ 6\ 2).$$

In this document we will always display permutations in this particular canonical form.

A.5 "Multiplying" Permutations

Each permutation represents a rearrangement of the objects in the slots and a very interesting idea is that you can combine two permutions by doing the first rearrangement and then rearranging those rearranged objects. If both permutations are represented in cycle notation, the calculation of the "product," meaning the result of appling one, then the other, is not hard to do.

Let's consider the rearrangement resulting from doing doing first $(6)(2\ 4)(1\ 3\ 5)$ followed by $(1\ 6)(2\ 3)(4\ 5)$.

To do the calculation, place the two permutations in order as follows:

$$(6)(2\ 4)(1\ 3\ 5)(1\ 6)(2\ 3)(4\ 5)$$

and work from left to right, following where the contents of each slot goes. For example, slot 1 goes to slot 3 and then slot 3 goes to slot 2 so the product begins:

$$(1\ 2 \ldots.$$

Start over, and 2 goes to slot 4 and then slot 4 goes to slot 5, so we have:

$$(1\ 2\ 5 \ldots.$$

Then we find that 5 goes to slot 6 in two steps and 6 back to 1 so the first cycle in the product is:

$$(1\ 2\ 5\ 6).$$

We notice that 3 has not been handled, and use the same process to find that the product of the two permutations is:

$$(1\ 2\ 5\ 6)(3\ 4).$$

In our canonical form, it becomes:

$$(3\ 4)(1\ 2\ 5\ 6).$$

It is worth noting that permutation multiplication is not commutative. In other words, if A and B are two permutations, it is not necessarily the case that $AB = BA$. In fact our example illustrates this. Check to see that:

$$(1\ 6)(2\ 3)(4\ 5)(6)(2\ 4)(1\ 3\ 5) = (2\ 5)(1\ 6\ 3\ 4).$$

B Group Theory

B.1 What is a Group?

A "group" is a mathematical object that consists of a set G of elements (groups can be finite or infinite, but in this article we will only consider finite groups) and a single binary operation $*$ on those elements satisfying the following four conditions:

1. The operation $*$ is closed. In other words, if $a \in G$ and $b \in G$ then $a * b \in G$.
2. The operation $*$ is associative. In other words, if a, b and c are any elements of G then:

   $$a * (b * c) = (a * b) * c.$$

3. There exists an identity element $e \in G$ such that for every $a \in G$:

   $$a * e = e * a = a.$$

4. For every $a \in G$ there exists an element $a^{-1} \in G$ called the inverse of a such that:

   $$a * a^{-1} = a^{-1} * a = e,$$

   where e is the identity element mentioned above.

Note that the operation $*$ is not necessarily commutative. There may be elements $a \in G$ and $b \in G$ such that $a * b \neq b * a$.

Since there is only one operation $*$ we often omit it and just place the group elements next to each other in the same way that we often omit the multiplication signs in algebra: $3xy$ is understood to be $3 \cdot x \cdot y$.

A group can be as simple as just the identity, where $e * e = e$ is the operation, or can be terribly complicated.

Although there are lots of examples of groups, in this article we will only be interested in permutation groups: groups such that the elements of G are permutations of objects in some set of slots. In such groups, if a and b are two permutations, then the permutation $a * b$ will simply be the "multiplication" of permutations as described in the previous appendix. The identity, e, in a permutation group is simply the permutation that leaves every object where it is. For a 5-element set $\{1, 2, 3, 4, 5\}$ of objects, we would have $e = (1)(2)(3)(4)(5)$.

The inverse of a permutation a is the permutation that undoes what a does. In other words, if a moves object in slot 1 to slot 3, then a^{-1} moves the object in slot 3 to slot 1. In the standard cycle notation, the inverse is obtained by reversing all the cycles. For example:

$$(1)(2\,3\,4)(5\,6)(7)(8\,9)^{-1} = (9\,8)(7)(6\,5)(4\,3\,2)(1).$$

B.1.1 S_3: The Symmetric Group on 3 Objects

Let's look in detail at a particular group:the group of all permutations of the three objects $\{1, 2, 3\}$. We know that there are $n!$ ways to rearrange n items since we can chose the final position of the first in n ways, leaving $n-1$ ways to chose the final position of the second, $n-2$ for the third, and so on. The product, $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$ is thus the total number of permutations. For three items that means there are $3! = 3 \cdot 2 \cdot 1 = 6$ permutations. Here is the list of the 6 in cycle notation:

$$(1)(2)(3),\ (3)(1\,2),\ (2)(1\,3),\ (1)(2\,3),\ (1\,2\,3),\ \text{and}\ (1\,3\,2).$$

Table 1 is a "multiplication table" for these six elements. Since, as we noted above, the multiplication is not necessarily commutative, the table is to be interpreted such that the first permutation in a product is chosen from the row on the top and the second from the column on the left. At the intersection of the row and column determined by these choices is the product of the permutations. For example, to multiply $(3)(1\,2)$ by $(2)(1\,3)$ choose the item in the second column and third row: $(1\,2\,3)$.

	$(1)(2)(3)$	$(3)(1\,2)$	$(2)(1\,3)$	$(1)(2\,3)$	$(1\,2\,3)$	$(1\,3\,2)$
$(1)(2)(3)$	$(1)(2)(3)$	$(3)(1\,2)$	$(2)(1\,3)$	$(1)(2\,3)$	$(1\,2\,3)$	$(1\,3\,2)$
$(3)(1\,2)$	$(3)(1\,2)$	$(1)(2)(3)$	$(1\,3\,2)$	$(1\,2\,3)$	$(1)(2\,3)$	$(2)(1\,3)$
$(2)(1\,3)$	$(2)(1\,3)$	$(1\,2\,3)$	$(1)(2)(3)$	$(1\,3\,2)$	$(3)(1\,2)$	$(1)(2\,3)$
$(1)(2\,3)$	$(1)(2\,3)$	$(1\,3\,2)$	$(1\,2\,3)$	$(1)(2)(3)$	$(2)(1\,3)$	$(3)(1\,2)$
$(1\,2\,3)$	$(1\,2\,3)$	$(2)(1\,3)$	$(1)(2\,3)$	$(3)(1\,2)$	$(1\,3\,2)$	$(1)(2)(3)$
$(1\,3\,2)$	$(1\,3\,2)$	$(1)(2\,3)$	$(3)(1\,2)$	$(2)(1\,3)$	$(1)(2)(3)$	$(1\,2\,3)$

Table 1: Multiplication of permutations of 3 objects

B.1.2 Subgroups

If G is a group, we can talk about a subgroup H of the group G. We say that H is a subgroup of G if the elements of H are a subset of the elements of G, and if we restrict the operation $*$ to H, then H itself will be a group.

If we let G be the symmetric group on three objects illustrated in Section B.1.1, we see that it has six subgroups: the entire group G itself, $\{(1)(2)(3)\}$, $\{(1)(2)(3),(3)(1\,2)\}$, $\{(1)(2)(3),(2)(1\,3)\}$, $\{(1)(2)(3),(1)(2\ 3)\}$, and $\{(1)(2)(3),(1\ 2\ 3),(1\ 3\ 2)\}$. Check that all of these are subgroups. The subgroups G itself and the trivial subgroup $\{(1)(2)(3)\}$ are not too interesting, but they are technically subgroups.

If you have a subset H of a group G and you want to show that H is a subgroup of G you can do so as follows:

- Show that the identity $e \in H$.
- Show that if a and b are any two elements of H then $ab \in H$.
- Show that if $a \in H$ then $a^{-1} \in H$.

You do not need to show associativity since the group operation is inherited from G and since G is a group the operation is automatically associative.

B.1.3 Cosets

If G is a group and H is a subgroup of G, then for any element $g \in G$, the "left coset" gH is defined to be the set:

$$gH = \{gh : h \in H\}.$$

(Note: the "right coset", Hg, is similarly defined, but we will not need it here. In the rest of this article, we will simply say "coset" instead of "left coset".)

Unless $g \in H$, the coset gH is not a group, but what is interesting is that the cosets have no overlap, they are all the same size, equal to the size of H, and that every element of G is in one of the cosets. This means that G can be divided into a number of chunks, each of which is the size of its subgroup H, which implies that the size of G is a perfect multiple of the size of H. Recall the examples in Section B.1.2: S_3 has 6 subgroups whose sizes are: 1, 2, 2, 2, 3, and 6: all of which are divisors of 6.

Using that group as an example again, here is a list of the left cosets of $H = \{(1),(1\,2)\}$:

$$\begin{aligned}
(1)(2)(3)H &= \{(1)(2)(3),(3)(1\,2)\} \\
(3)(1\,2)H &= \{(1)(2)(3),(3)(1\,2)\} \\
(2)(1\,3)H &= \{(2)(1\,3),(1\,3\,2)\} \\
(1\,3\,2)H &= \{(2)(1\,3),(1\,3\,2)\} \\
(1)(2\,3)H &= \{(1)(2\,3),(1\,2\,3)\} \\
(1\,2\,3)H &= \{(1)(2\,3),(1\,2\,3)\}
\end{aligned}$$

Notice that the cosets are all of size 2, which is the size of H, and that they are either identical or completely disjoint. We will show that this is the case for any group G having subgroup H, namely:

1. Every coset has the same number of elements as H.
2. Every element of G is in some coset.
3. If two cosets share a single element, then they are identical.

Here are the proofs for all three statements above:

1. First, let's show that all cosets are the same size as the subgroup H. The only way that a coset gH can have fewer elements is if $gh_1 = gh_2$ where $h_1 \neq h_2$. If $gh_1 = gh_2$ then $g^{-1}gh_1 = g^{-1}gh_2$ so $eh_1 = eh_2$, so $h_1 = h_2$. Thus all cosets are the same size as the subgroup H.

2. Obviously every element $g \in G$ is in some coset, since $g \in gH$ because $e \in H$, and $ge = g \in gH$.

3. Now we will show that if the cosets g_1H and g_2H share any single element $g \in G$ then they are identical. Let $g \in g_1H$ and $g \in g_2H$. Then $g = g_1h_1$ and $g = g_2h_2$ for some $h_1, h_2 \in H$. To show that $g_1H = g_2H$ we need to show that if $g_3 \in g_1H$, then $g_3 \in g_2H$.

   If $g_3 \in g_1H$, then $g_3 = g_1h_3$, for some $h_3 \in H$. Since $g = g_1h_1$ and $g = g_2h_2$ we have:

   $$g_1h_1 = g_2h_2.$$

   Multiply both sides above on the right by h_1^{-1} to obtain:

   $$g_1h_1h_1^{-1} = g_2h_2h_1^{-1}$$

   or

   $$g_1 = g_2h_2h_1^{-1}.$$

   Now we know that $g_3 = g_1h_3$ and we can substitute the value of g_1 in the equation above to obtain:

   $$g_3 = g_2h_2h_1^{-1}h_3.$$

   But since H is a subgroup (and hence a group) $h_4 = h_2h_1^{-1}h_3 \in H$ and we have:

   $$g_3 = g_2h_4,$$

   so g_3 is in the coset g_2H and we are done.

Note: Although we don't need this fact in this document, the result above shows that if G is a finite group and if H is a subgroup of G, then the size of H divides evenly into the size of G. This is because the left cosets of H are all the same size and every element of G is in exactly one of them, so the cosets divide the members of the group G into a bunch of equal-sized cosets.

C Solutions

C.1 A Square Tablecloth

In how many ways can a square tablecloth that is divided into 5×5 squares be colored with k colors? There are two answers, depending on whether the tablecloth can be flipped over and rotated or simply rotated to make equivalent patterns.

There will be 25 squares (slots) and let's assign names to the slots as follows. Note that if we allow rotations and reflections, a_i slots can only go to a_i slots, and so on.

a_1	b_1	c_1	b_2	a_2
b_8	d_1	e_1	d_2	b_3
c_4	e_4	f_1	e_2	c_2
b_7	d_4	e_3	d_3	b_4
a_4	b_6	c_3	b_5	a_3

If we allow only rotations of the tablecloth there are only four possible rotations:

$$(a_1)(a_2)(a_3)(a_4)(b_1)(b_2)(b_3)(b_4)(b_5)(b_6)(b_7)(b_8)(c_1)(c_2)(c_3)(c_4)(d_1)(d_2)(d_3)(d_4)(e_1)(e_2)(e_3)(e_4)(f_1)$$

$$(a_1a_2a_3a_4)(b_1b_3b_5b_7)(b_2b_4b_6b_8)(c_1c_2c_3c_4)(d_1d_2d_3d_4)(e_1e_2e_3e_4)(f_1)$$

$$(a_1a_3)(a_2a_4)(b_1b_5)(b_3b_7)(b_2b_6)(b_4b_8)(c_1c_3)(c_2c_4)(d_1d_3)(d_2d_4)(e_1e_3)(e_2e_4)(f_1)$$

$$(a_1a_4a_3a_2)(b_1b_7b_5b_3)(b_2b_8b_6b_4)(c_1c_4c_3c_2)(d_1d_4d_3d_2)(e_1e_4e_3e_2)(f_1)$$

These are not too surprising; it's just six groups of slots that rotate around and one slot in the center that stays fixed.

If we add the four reflections about the center lines and the diagonals, we add the following four permutations:

$$(c_1)(e_1)(f_1)(e_3)(c_3)(a_1a_2)(b_1b_2)(b_8b_3)(d_1d_2)(c_4c_2)(e_4e_2)(b_7b_4)(d_4d_3)(a_4a_3)(b_6b_5)$$

$$(c_4)(e_4)(f_1)(e_2)(c_2)(a_1a_4)(b_1b_6)(c_1c_3)(b_2b_5)(a_2a_3)(b_8b_7)(d_1d_4)(e_1e_3)(d_2d_3)(b_3b_4)$$

$$(a_1)(d_1)(f_1)(d_3)(a_3)(b_1b_8)(c_1c_4)(b_2b_7)(a_2a_4)(e_1e_4)(d_2d_4)(b_3b_6)(e_2e_3)(c_2c_3)(b_4b_5)$$

$$(a_2)(d_2)(f_1)(d_4)(a_4)(a_1a_3)(b_1b_4)(c_1c_2)(b_2b_3)(b_8b_5)(d_1d_3)(e_1e_2)(c_4c_3)(e_4e_3)(b_7b_6)$$

Allowing only rotations, the cycle index polynomial will be:

$$P_1 = \frac{f_1^{25} + 2f_1^1 f_4^6 + f_1^1 f_2^{12}}{4}.$$

If we include refections we obtain:

$$P_2 = \frac{f_1^{25} + 2f_1^1 f_4^6 + f_1^1 f_2^{12} + 4f_1^5 f_2^{10}}{8}.$$

With k colors and counting all possible patterns (in other words, letting all the variables be set to 1) we obtain:

$$P_1 = \frac{k^{25} + 2k^7 + k^{13}}{4}$$

and

$$P_2 = \frac{k^{25} + 2k^7 + k^{13} + 4k^{15}}{8}.$$

C.2 Faces of a Cube

How many ways can you color the six faces of a cube such that 1 is colored red, 2 are green, and 3 are blue? How many total red-green-blue colorings of the cube are there?

We could work out all the permutations of the cube faces in detail, but we don't need to know exactly what they are; we just need to know the structure of the permutations to construct the inventory polynomial and with that we can obtain the answers to the questions above.

It is fairly easy to see that there are exactly 24 rotations of the cube in three-dimensional space: Each face can be mapped to any of the six faces, and there are four possible rotations about that face, for a total of $6 \cdot 4 = 24$ permutations. Those permutations can be divided into four classes, and the cycle structure of each class is easy to work out. Here they are, where all that matters is the form of the permutations. It doesn't matter what the particular faces are, so they are just numbered from 1 to 6 below:

- The identity. (1)(2)(3)(4)(5)(6).
- Rotation about the center of a face. There are 6 faces so there are three axes where a rotation can be applied. There are three possible rotations: $90°$, $180°$, and $270°$. (The rotation of $0°$ is the identity which we have already counted.) There are thus 9 permutations.

  Since two opposite faces are fixed in these rotations, each permutation will begin with cycles that look like this: $(1)(2)$. The other four faces just rotate like the beads on a necklace, so two of them (for $90°$ and $270°$) will have a final form like $(1)(2)(3\ 4\ 5\ 6)$. There are six of these. The $180°$ rotation yields something of the form $(1)(2)(3\ 4)(5\ 6)$. There are three of these.
- Imagine an axis through the center of opposite edges of the cube. A rotation by $180°$ will swap three pairs of faces, yielding permutations of the form $(1\ 2)(3\ 4)(5\ 6)$. Since there are 12 edges, there are 6 opposite pairs, and thus six permutations of this form.
- Finally, look at rotations about the long diagonals connecting the vertices of the cube. There are three faces that meet at each vertex so there are rotations of $0°$, $120°$ and $240°$. The rotation by $0°$ is just the identity, so only the second two yield valid rotations. There are 8 vertices, so 4 pairs, and each such rotation moves opposite pairs of cube faces in a cycle of length three. Thus there are eight permutations of the form: $(1\ 2\ 3)(4\ 5\ 6)$.

That makes 24, so we have them all. Here is a list of the cycle structures with a count of each type:

$$\begin{array}{rcl} 1 & : & (1)(2)(3)(4)(5)(6) \\ 6 & : & (1)(2)(3\ 4\ 5\ 6) \\ 3 & : & (1)(2)(3\ 4)(5\ 6) \\ 6 & : & (1\ 2)(3\ 4)(5\ 6) \\ 8 & : & (1\ 2\ 3)(4\ 5\ 6) \end{array}$$

Here is the cycle index polynomial:

$$P = \frac{f_1^6 + 6f_1^2 f_4^1 + 3f_1^2 f_2^2 + 6f_2^3 + 8f_3^2}{24}.$$

It is easy to find the total number of red-green-blue colorings there are. Just substitute 3 (the number of colors) for each of the f_i to obtain:

$$\frac{3^6 + 6 \cdot 3^2 \cdot 3 + 3 \cdot 3^2 \cdot 3^2 + 6 \cdot 3^3 + 8 \cdot 3^2}{24} = 57.$$

To find the number of colorings with one red, two green, and three blue we need to substitute the appropriate polynomials in x, y, and z for the f_i and pick out the number of terms having the form xy^2z^3 where x is the "color" of red, et cetera.

Substituting, we obtain:

$$P = \frac{\begin{array}{c}(x+y+z)^6 + 6(x+y+z)^2(x^4+y^4+z^4) + 3(x+y+z)^2(x^2+y^2+z^2)^2 \\ +6(x^2+y^2+z^2)^3 + 8(x^3+y^3+z^3)^2\end{array}}{24}.$$

We need to find the terms above that might generate an xy^2z^3. The first one will, but the second one can't since it will have some fourth power in every term. The third one will, but only if we get something like xz from the first factor and y^2z^2 from the second. The last two terms obviously contribute nothing either.

For the first term, the coefficient comes from the multinomial $6!/(3!2!1!) = 60$. In the third term, the xz in the first factor has a coefficient of 2 as does the y^2z^2 part of the second factor. The leading 3 gives $3 \cdot 2 \cdot 2 = 12$. Thus the factor of xy^2z^3 will be $(60+12)/24 = 3$, so there are three such colorings. See if you can figure out what they are by hand. It's not too hard.

C.3 Faces of a Dodecahedron

How many ways can you color the faces of a reular dodecahedron with 5 different colors? How many ways can you color them with red and four other colors where exactly 5 of the faces are colored red and the other faces can be colored arbitrarily?

Figure 3 shows the front 6 faces of a regular dodecahedron. The back looks exactly the same with each face in the back parallel to one of the faces in the front.

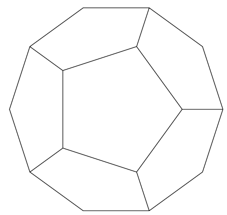


Figure 3: Dodecahedron

A regular dodecahedron has twelve pentagonal faces, three of which meet at each vertex. There are 60 symmetries since a particular face can be mapped to any other face, and can be mapped to that other face rotated to any of five positions. Thus there are $12 \cdot 5 = 60$ rotational symmetries.

Using the same idea as with the cube symmetries, we will not need to find the exact permutations; all we need to do is find the cycle structure of the symmetries. Here they are:

- The identity that maps all 12 faces to themselves. The form of this symmetry is just $(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)$.
- Rotations around the axis connecting the six pairs of opposite faces. In every case, four of those rotations are new, so there are $6 \cdot 4 = 24$ of these. The two faces that determine the axis of rotation map to themselves, and the ten other faces are rotated in two groups of five, so every one of these 24 rotations has the form: $(1)(2)(3\ 4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12)$.
- Rotations about the axes through opposite vertices. The dodecahedron has 20 vertices and therefor 10 such axes. Since three pentagons meet at each vertex, there are three symmetric positions, one of which is the identity, so each axis determines two new rotations, for a total of $10 \cdot 2 = 20$ symmetries. None of them fix any faces the faces are rotated in four sets of three. Thus the structure of these permutations is: $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$.
- Finally, there are rotations of $180°$ about axes determined by opposite pairs of edges. There are 30 edges and so 15 pairs. No faces are fixed, so each rotation swaps six pairs of faces, making these permutations have the form: $(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)$.

This accounts for $1 + 24 + 20 + 15 = 60$ permutations so we have them all. Here is a list of the cycle structures with a count of each type:

$$\begin{array}{rcl}
1 & : & (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12) \\
24 & : & (1)(2)(3\ 4\ 5\ 6\ 7)(8\ 9\ 10\ 11\ 12) \\
20 & : & (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12) \\
15 & : & (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)
\end{array}$$

Here is the cycle index polynomial:

$$P = \frac{f_1^{12} + 24 f_1^2 f_5^2 + 20 f_3^4 + 15 f_2^6}{60}.$$

For the total number of ways with 5 colors, we obtain, in the same way as before:

$$\frac{5^{12} + 24 \cdot 5^4 + 20 \cdot 5^4 + 15 \cdot 5^6}{60} = 4073375.$$

To solve the second part of the problem: 5 red and the other seven faces arbitrarily colored with four other colors, one way to do it would be to expand the cycle index polynomial substituted with terms of the form $(x^i + y + i + z^i + w^i + v^i)$ with appropriate i values, expand the mess, and then pick out all the terms with x^5 in them and add their coefficients.

But it is easier just to let all the variables except for x be 1, do the expansion, and pick out the term with x^5. Here is the unexpanded form:

$$\frac{((x+4)^{12} + 24(x+4)^2(x^5+4)^2 + 20(x^3+4)^4 + 15(x^2+4)^6)}{60},$$

which after expansion becomes:

$$\begin{aligned} &280832 + 838912x + 1154976x^2 + 961280x^3 + 541632x^4 + 216320x^5 \\ &\quad +63456x^6 + 13520x^7 + 2172x^8 + 240x^9 + 30x^{10} + 4x^{11} + x^{12}. \end{aligned}$$

The answer to our question is thus 216320. Notice that this also solves the problem for any specific number of red faces where all the others are arbitrarily colored with four other colors. A quick sanity check on our answer is provided by the final two terms. The coefficient of x^{12} is 1, since if all twelve faces are colored red, there's obviously only one way to do it. If eleven are colored red, the final one is one of four colors, so the coefficient 4 is obviously correct.

We can even check the 30 if all but two are red. The two non-reds can be opposite each other, adjacent to each other, or neither opposite nor adjacent, so there are three ways to pick the spots for the two non-red faces. There are 4 ways to do it if the two colors are the same, and 6 ways to pick two different colors from a set of four. So there are $4 + 6 = 10$ ways to choose the two non-red faces and for each of those choices, three ways to place the two colors for a grand total of $3 \cdot 10 = 30$.

The other coefficients are harder to check, but there is one more number that we can calculate, relatively easily. That's the one with no x term: 280832. This must be the number of ways to color the dodecahedron with only four colors, and we can compute that directly by substituting 4 for 5 in the formula we used to count all colorings with five. Here is the substitution and the correct result:

$$\frac{4^{12} + 24 \cdot 4^4 + 20 \cdot 4^4 + 15 \cdot 4^6}{60} = 280832.$$

C.4 Necklaces

In how many ways can a necklace with 12 beads be made with 4 red beads, 3 green beads, and 5 blue beads? How many necklaces are possible with n beads of k different colors? (Depending on the type of beads, some necklaces can be turned over and some cannot, so there are really four different problems here.)

To solve the n beads and k colors part we need to look at the structure of the so-called cyclic groups (if the necklaces cannot be turned over) or dihedral groups (if turning over is an option).

C.4.1 Cyclic Groups

A cyclic group is a group that can be generated by a single permutation. In our case such a single permutation is the one that advances every element on the necklace to the next clockwise spot. If we repeat this operation enough times we will obviously generate all possible rotations.

Let's consider the 12-bead necklace in the first part of the problem. Here are the twelve rotations expressed in cycle form where we assume that the slots are numbered from 1 to 12 in clockwise order. The first is the identity (e: no rotation) and the second is the generator g—a rotation by a single position which, when repeated, generates all the elements of the group:

$$\begin{aligned}
e = g^0 &= (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)\\
g^1 &= (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)\\
g^2 &= (1\ 3\ 5\ 7\ 9\ 11)(2\ 4\ 6\ 8\ 10\ 12)\\
g^3 &= (1\ 4\ 7\ 10)(2\ 5\ 8\ 11)(3\ 6\ 9\ 12)\\
g^4 &= (1\ 5\ 9)(2\ 6\ 10)(3\ 7\ 11)(4\ 8\ 12)\\
g^5 &= (1\ 6\ 11\ 4\ 9\ 2\ 7\ 12\ 5\ 10\ 3\ 8)\\
g^6 &= (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12)\\
g^7 &= (1\ 8\ 3\ 10\ 5\ 12\ 7\ 2\ 9\ 4\ 11\ 6)\\
g^8 &= (1\ 9\ 5)(2\ 10\ 6)(3\ 11\ 7)(4\ 8\ 12)\\
g^9 &= (1\ 10\ 7\ 4)(2\ 11\ 8\ 5)(3\ 12\ 9\ 6)\\
g^{10} &= (1\ 11\ 9\ 7\ 5\ 3)(2\ 12\ 10\ 8\ 6\ 4)\\
g^{11} &= (1\ 12\ 11\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2)
\end{aligned}$$

Notice that every element above has a similar structure: each is made with some number of cycles of equal length. Since all the elements except for the identity move all the slot contents we know that the cycle lengths have to divide evenly into 12 and for every permutation above the product of the cycle length and the number of cycles must be 12.

Note that if the exponent of g is relatively prime to 12 then, and only then, is the permutation made of a single cycle of length 12. The numbers that have no common factors with 12 except for 1 are: $\{1, 5, 7, 11\}$. In fact, here is a list of the exponents that generate permutations having all cycles of certain lengths together with the greatest common divisors of those exponents and 12:

Cycle Length	Permutations	GCD with 12
1	g^0	$GCD(0,12)=12$
2	g^6	$GCD(6,12)=6$
3	g^4, g^8	$GCD(4,12)=GCD(8,12)=4$
4	g^3, g^9	$GCD(3,12)=GCD(9,12)=3$
6	g^2, g^{10}	$GCD(2,12)=GCD(10,12)=2$
12	g^1, g^5, g^7, g^{11}	$GCD(1,12)=GCD(5,12)=GCD(7,12)=GCD(11,12)=1$

Notice that the GCD's above are the number of cycles and if you divide that into 12, you obtain the length of each cycle. There is nothing special about 12, either.

Now we have a way to work out the cycle structure of any cyclic group of permutations. For 12, the cycle index polynomial will be:

$$P_{cyc} = \frac{f_1^{12} + f_2^6 + 2f_3^4 + 2f_4^3 + 2f_6^2 + 4f_{12}^1}{12}.$$

If we can't turn the necklace over and we are trying to find the number of possible colorings with 4 red, 3 green, and 5 blue beads, we can substitute $(x^i + y + i + z^i)$ for f_i above and search the expansion for terms of the form $x^4y^3z^5$.

An examination of the terms shows that the only one that can possibly generate terms of this type is the f_1^{12} which will be replaced by $(x + y + z)^12$. We need the multinomial coefficient:

$$\frac{12!}{4!3!5!} = 27720$$

and if we divide that by 12 we obtain 2310 different colorings.

To count all colorings with 3 colors, just work out:

$$\frac{3^{12} + 3^6 + 2 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 4 \cdot 3}{12} = 44368.$$

C.4.2 Dihedral Groups

If the necklace can be turned over there will the twice as many permutations given that the flipped necklace can be rotated to any of the necklace's positions. It turns out, however, that the form of these is easy: if there are an odd number, every reflection through a bead and the center of the necklace yields one, so if the odd number has the form $2m + 1$ there will be one 1-cycle and $2m$ with m of the 2-cycles. If there are an even number of beads (say $2m$), we can reflect through opposite pairs of beads for half of them, and reflect through opposite pieces of string between a pair of beads for the others. Thus there will be m permutations with m of the 2-cycles and m permutations with two 1-cycles and $m - 1$ with $m - 1$ of the 2-cycles.

For our example with 12 beads, that adds two terms to the numerator of our previous P and doubles the denominator from 12 to 24:

$$P_{dih} = \frac{f_1^{12} + f_2^6 + 2f_3^4 + 2f_4^3 + 2f_6^2 + 4f_{12}^1 + 6f_1^2f_2^5 + 6f_2^6}{24}.$$

The only term that matters for the $4, 3, 5$ problem is the $6f_1^2f_2^5$ and this will yield 360 more copies of $x^4y^3z^5$, so we add that to 27720 but now need to divide by 24 yielding 1170 valid colorings. It makes sense that the number of patterns is reduced since some of the ones we had previously match others if one is flipped over and rotated appropriately.

To obtain the total number available with three colors, we need to expand this:

$$\frac{3^{12} + 3^6 + 2 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 4 \cdot 3 + 6 \cdot 3^7 + 6 \cdot 3^6}{24} = 22913.$$