Subdividing a Pile

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Abstract

This is a beautiful problem that can be used for middle school students and older. On the face of it, the problem is interesting, but students working on it are actually drilling their multiplication and addition facts.

1 Introduction

This document is meant for the teacher. It describes an interesting problem and then talks about various ways the problem can be used in a classroom. Depending on the age and sophistication of the students, the classroom discussion can be taken in different directions.

If you are the teacher, probably the best way to use this document is first to read the problem in the next section and before you read on, you should try it yourself to get an idea of how the students might be thinking and to think about how you might use it in a class. Then read on to see our ideas.

2 The Problem

Start with a single pile of chips. We will eventually look at different sized initial piles, but for now, let's begin with a pile containing exactly 10 chips.

At each stage, you can choose a pile and split it into two piles, but those two piles do not need to be equal or even roughly equal. You could divide the initial pile of 10 into two piles containing 5 each, or you could split it into a pile containing 9 and another pile containing only 1 chip. Continue until all the piles have size 1 which must happen eventually, since piles keep getting smaller, but none can contain fewer than 1 chip.

Your score is determined as follows: Your initial score is zero, and each time you divide a pile into two, you add to your score the product of the sizes of the two new piles.

The result of a sample game is listed below. Each row corresponds to a stage in the game where the first row is the starting position. The column entitled "Piles" lists the current set of piles, and the pile in bold face is the one that was chosen to be split. The "Score" column lists your running score and indicates how it is computed, and the "Move" column displays the move that you make from that position (which will generate the position at the beginning of the next row. A move like "$4 \rightarrow 3, 1$" means "split a pile containing 4 items into a pile containing 3 and another containing 1":

Piles	Score	Move
10	0	$10 \to 6, 4$
6, 4	$0 + 6 \times 4 = 24$	$6 \to 3, 3$
3, $3, 4$	$24 + 3 \times 3 = 33$	$3 \to 2, 1$
$2, 1, 3,$ **4**	$33 + 2 \times 1 = 35$	$4 \to 2, 2$
$2, 1, 3,$ **2**, 2	$35 + 2 \times 2 = 39$	$2 \to 1, 1$
2, $1, 3, 1, 1, 2$	$39 + 1 \times 1 = 40$	$2 \to 1, 1$
$1, 1, 1,$ **3**, $1, 1, 2$	$40 + 1 \times 1 = 41$	$3 \to 2, 1$
$1, 1, 1, 2, 1, 1, 1,$ **2**	$41 + 2 \times 1 = 43$	$2 \to 1, 1$
$1, 1, 1,$ **2**, $1, 1, 1, 1, 1$	$43 + 1 \times 1 = 44$	$2 \to 1, 1$
$1, 1, 1, 1, 1, 1, 1, 1, 1, 1$	$44 + 1 \times 1 = 45$	Game over

For this example with 10 chips, how much how much higher a score can you get than 45? What is the smallest possible score? Can you solve problems with different numbers of chips in the initial pile? Are there general formulas for the best and worst possible scores that can be obtained beginning with a single pile containing n chips?

3 Classroom Presentation

The students can work on this problem alone or perhaps in teams of two. It is probably a good idea to provide each student or team with a stack of chips. Before you hand out the chips, go through one example (like the one at the end of the previous section) in detail, physically splitting your pile and keeping a list of the running scores, at least. You don't need to list all the existing piles, but if you write down something as simple as "3×2" to represent a move, you know that the move must have been $5 \to 3, 2$ so that if you were listing the piles, you'd know that a 5 turned into a 3 and a 2.

Ask the students to come up with the highest-scoring and lowest-scoring games that they can. After a few minutes, ask who has a higher or lower score than the one you got in your sample subdivision. There will probably be a lot of arithmetic errors, and (as you will see) it is easy to know when an error has occurred, so you can point out the error and have the students re-evalute their score.

Finally, the class should be convinced of the result, and then you can begin to lead them in explorations of why the result might be true.

Another question you might pose is this: "How many steps will the game take?". The answer is that (starting with 10 chips) it will take exactly 9 steps, no matter how the game is played. That's because you begin with one pile and each stage generates exactly one more pile. Since the game ends when there are 10 piles, it must last exactly 9 steps. So a game beginning with a single pile of n chips will be over after $n - 1$ moves.

4 The Result

What the students should find is that the score is always *exactly the same*. If you begin with 10 chips, you will always have a score of 45, no matter how the subdivision is done. In fact, if you begin with n chips in a single pile, the score will always be $n(n - 1)/2$.

Thus if any student or team comes up with a number that is not 45, you know that at least one arithmetic error has occurred. Make them show you their calculations, and you will always find an error.

5 How to Investigate the Problem

A strategy that works well in situations like this is to investigate similar, simpler problems. You can then make tables of the results and patterns begin to emerge. If you ask the students for a simpler problem, many will suggest a smaller initial pile, but they will probably suggest "smaller" numbers like 4 or 5. Tell them that it's almost always best to start with the very smallest, even if it seems totally trivial, since it may help to find the general pattern. In your class, keep adding to this table as you gather more information, and don't erase it so that you can always refer back to the results of previous work.

In this case, first look at the situation where the initial pile contains only a single chip. The game is obviously over instantly, with a score of zero (since no subdivisions are possible, and the only way to add to your score is to do a subdivision). So you can begin writing a table on the board that looks something like this:

Initial Pile Size	Score
1	0

The next simplest game obviously begins with a pile containing 2 chips. There is only one move: $2 \to 1, 1$, giving you a grand score of $1 \times 1 = 1$, so your table on the blackboard now looks like this:

Initial Pile Size	Score
1	0
2	1

There is no problem with an initial pile size of 3, either, since the only possible initial move is $3 \to 2, 1$ (for a score of $2 \times 1 = 2$), after which you need to subdivide the pile containing 2 items (adding $1 \times 1 = 1$ to the score), and the total score is $2 + 1 = 3$.

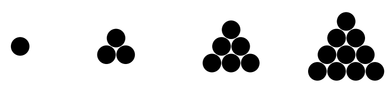
The first "interesting" position begins with a pile of 4 chips, and it is interesting because there is more than one possible move: $4 \to 1, 3$ or $4 \to 2, 2$. After the initial move everything is determined, however, and it's pretty easy to show that the score is 6 in either case.

For an initial position with 5, again there are two possible moves: $5 \to 1, 4$ and $5 \to 2, 3$. As you start to work on this, point out to the students that if your first move was $5 \to 1, 4$, they don't need to work out the score for 4 since they already did it for the previous problem. If you begin with $5 \to 1, 4$, their final score will include the $1 \times 4 = 4$ from this move plus the score obtained by subdividing the pile that contains 4 items, and they already know that they'll get 6 from that, yielding a final score of 10. Similarly, if the first move is $5 \to 2, 3$, they get a score of $2 \times 3 = 6$ for the first move and they simply have to add on the results from an initial pile of size 2 (a score of 1) and from an initial pile of size 3 (a score of 3) for a total of $6 + 1 + 3 = 10$. In both cases, the final score is 10, so the table now looks like this:

Initial Pile Size	Score
1	0
2	1
3	3
4	6
5	10

It might be worth working through the next stage with the class beginning with an initial pile of 6 since this time there are three possibilities for the first move: $6 \to 1, 5$, $6 \to 2, 4$ and $6 \to 3, 3$. Each value can be determined in a single step from the previously computed values, which will be, respectively, $1 \times 5 + 10 = 15$, $2 \times 4 + 1 + 6 = 15$ and $3 \times 3 + 3 + 3 = 15$, so the pair $(6, 15)$ can be added to the table above.

Depending on the sophistication of the class, the list of numbers above may remind them of the triangular numbers that count the number of items required to make a triangle of different sizes. The figure below illustrates the first four triangular numbers:



The number of dots to make the triangles above is obviously 1, $1+2 = 3$, $1+2+3 = 6$, $1+2+3+4 = 10$, et cetera. These are exactly the same numbers we have in our table.

At this point, the class may be willing to bet that the answer is constant, and if that is the case, why might the answer be the triangular numbers? Well, look at a very simple way of subdividing piles. Suppose we start with a pile of 6 and do the following subdivisions: $6 \to 1, 5$, $5 \to 1, 4$, $4 \to 1, 3$, $3 \to 1, 2$ and finally, $2 \to 1, 1$. The score will be: $5 + 4 + 3 + 2 + 1 = 15$; obviously a triangular number, and the same technique will yield a triangular number for any initial position.

Of course we have not *proved* that we always obtain the same number, but we have certainly looked at a lot of examples.

6 Proof Using Mathematical Induction

This is *not* the easiest way to solve the problem; that will be presented in Section 7, but this method is probably the first one that would occur to a mathematician, since it provides a straight-forward, brute-force solution, assuming you are adept at algebra.

If the class is more advanced and the students know something about mathematical induction, here is a straight-forward proof that the answer is constant, and that in fact for an initial pile containing n chips, the final score will be the triangular number $n(n-1)/2$. If the class knows nothing about mathematical induction, skip to the next section for a much more intuitive proof.

In fact, what is required here is *strong* mathematical induction. Usually a proof by mathematical induction proves the result for the simplest case (in our example this would be for an initial pile containing 1 chip) and then shows that if the result is true for the case of $k-1$ chips, it is true for the case of k chips[1]. For strong induction, we again begin by proving it for the 1 chip case, but to prove it for the case with k chips, we assume that it is true for *all* values below k; not just for $k-1$.

In our case, if $n=1$, there is one initial chip and no possible moves, so the final score is zero, and if $n=1$, the value of $n(n-1)/2$ is 0, so we are done.

Now assume that for every pile of size m, where $m<k$ the score obtained by subdividing that pile in any order is given by the formula $m(m-1)/2$. Suppose that an initial pile containing k chips is subdivided into m and $k-m$ chips, where $0<m<k$. The score will be $m(k-m)$ added to the scores obtained by subdividing piles of size m and $k-m$. By the induction hypothesis, those subdivisions will yield scores of $m(m-1)/2$ and $(k-m)(k-m-1)/2$, respectively, no matter how the subdivision is done. Thus the final score S for a subdivision that begins with $k \to m, (k-m)$ is:

$$S = m(k-m) + \frac{m(m-1)}{2} + \frac{(k-m)(k-m-1)}{2}.$$

A little algebra yields:

$$\begin{aligned}
S &= m(k-m) + \frac{m(m-1)}{2} + \frac{(k-m)(k-m-1)}{2} \\
S &= \frac{2m(k-m) + m(m-1) + (k-m)(k-m-1)}{2} \\
S &= \frac{2mk - 2m^2 + m^2 - m + k^2 - 2mk + m^2 - k + m}{2} \\
S &= \frac{k^2-k}{2} = \frac{k(k-1)}{2}.
\end{aligned}$$

The final result is exactly what we expected, and does not depend at all on the choice of m for the first subdivision.

The proof above is rock-solid, but it is a little disappointing, since it doesn't really tell us much about what is going on.

Even for students who have been exposed to mathematical induction, the proof above may be a little frightening since there are two variables: m and k, involved. In most examples that students have seen, there is a single variable such as k that is somehow converted to $k+1$ in the next stage. It is probably worth looking at what the algreba above says for a specific case that you have already checked out. For example, show how the result for $k=6$ is derived if $m=2$, meaning that the initial pile of $k=6$ is split into $m=2$ piles and $k-m=4$ piles.

In the first line, $m(k-m) = 2(4) = 8$ is the score obtained from the split. From the earlier induction steps, we know that a pile containing $m=2$ will add $m(m-1)/2 = 2(1)/2 = 1$ to the score and a pile containing $(k-m)=4$ chips will add $(k-m)(k-m-1)/2 = 4(3)/2 = 6$ to the score. The resulting score for this case is $8+1+6=15$, which is equal to $k(k-1)/2 = 6(5)/2$.

[1] Or equivalently, we assume it is true for k and show it is true for $k+1$.

7 Counting Handshakes

If there are m people in a room, and each shakes hands exactly once with every other person in the room, how many total handshakes are there? For a class, you probably want to look at some specific numbers, rather than just n, but it is pretty easy to see what is going on. Suppose there are 5 people. The first one will have to shake 4 hands. The second person has already shaken hands with the first, so needs only shake hands with the 3 remaining people, and thus there are $4+3$ handshakes so far. The third person has already shaken hands with the first and second, and needs to shake only 2 hands to complete the set. The fourth person similarly only needs to shake one hand, and the last person has already done. Thus there are $4+3+2+1=10$ total handshakes, and the same argument will clearly work to show that for n people, the number of handshakes is $(n-1)+(n-2)+(n-3)+\cdots+2+1$, which is the triangular number whose value is $n(n-1)/2$ (see Section 8).

Instead of a pile of chips, imagine that we are subdividing a "pile" of people. Initially, the people all shake hands, but as they do so, they tie a long piece of string between the wrists of any pair that has shaken hands. From the paragraph above, there are obviously $n(n-1)/2$ total strings.

Now divide the group into two piles. There will be a bunch of strings that run between the two piles, and to really separate them, we will need to cut all the connecting strings. If there are k people in one pile and m in the other, there will be km strings between the piles (each of the k people will have m strings connecting him or her to the m people in the other pile with whom he or she has shaken hands). Thus you will need to cut mk strings.

The same reasoning can be applied to any pile subdivision: the number of strings cut with each subdivision is equal to the product of the number of people in the two newly-created piles. When subdivision is complete, *all* the strings are cut, and therefore the total of all the products must be the same as the initial number of strings: $n(n-1)/2$.

This completes the proof.

8 Formula for the Triangular Numbers

In this section we will prove in two different ways that:

$$1+2+3+\cdots+(n-1)=\frac{n(n-1)}{2}.$$

As usual, if you are trying to convince your students, don't start with the sum above, but with a concrete example; say, add the numbers from 1 to 7, the result of which you can easily check by hand: $1+2+\cdots+7=28$. (Note that this corresponds to the case $n=8$, since we are only summing the numbers from 1 to $n-1$).

If the (so far unknown sum) is S, we can write:

$$S=1+2+3+4+5+6+7.$$

Since the order of addition doesn't matter, we could write S equally well as:

$$S=7+6+5+4+3+2+1.$$

If we add the two equations above, we obtain:

$$\begin{aligned}
S &= 1+2+3+4+5+6+7 \\
\underline{S} &= \underline{7+6+5+4+3+2+1} \\
2S &= 8+8+8+8+8+8+8 \;=\; 7\times 8 \;=\; 56 \\
2S &= 56 \\
S &= 56/2 = 28.
\end{aligned}$$

It should be clear that there's nothing special about adding 7 numbers. If S is the sum from 1 to $n-1$, then S can also be written as the sum from $n-1$ to 1 (in reverse order). We can add the two equations and find that $2S$ will consist of $n-1$ copies of n, so the sum is given by half of $n(n-1)$:

$$S = 1+2+3+\cdots+(n-2)+(n-1) = n(n-1)/2. \tag{1}$$

Note: the usual way that the sum of consecutive integers is written is as:

$$1+2+3+\cdots+(n-1)+n = (n+1)n/2. \tag{2}$$

We only need to add from 1 to $n-1$ (since the first person in a group of n needs to make $n-1$ handshakes, the next person $n-2$, and so on). Note that Equation 1 can be obtained from Equation 2 by substituting $n-1$ for n.

An equally valid proof that $1+2+3+\cdots+(n-1) = n(n-1)/2$ can be obtained by simply counting the handshakes we used to visualize the situation in Section 7. Each person shakes hands with all the other people, so if there are n people, each of them shakes hands with $n-1$ people yielding what at first glance appears to be $n(n-1)$ total handshakes. Notice, however, that this double-counts the handshakes: we counted both the case where John shook Mary's hand and Mary shook John's. Thus $n(n-1)$ counts each handshake twice, so the actual number is $n(n-1)/2$.