

# Iterated Functions

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## Abstract

In this article we will examine various properties of iterated functions. If  $f(x)$  is a function, then the iterates of  $f$  are:  $f(x), f(f(x)), f(f(f(x))), \dots$

## 1 Introduction

I first became fascinated by iterated functions when I had a scientific calculator for the first time and repeatedly pressed the “cosine” button.

The calculator was in “radians” mode, so the angles were interpreted in radians rather than degrees<sup>1</sup>, but I found that no matter what number I started with, after enough button presses of the “cosine” button, the numbers seemed to approach .739085133215.

What I was doing, of course, was iterating the cosine function. If my starting number was 1, then pressing the cosine button repeatedly was generating the following sequence of numbers:

$$\begin{aligned}\cos(1) &= .540302305868 \\ \cos(\cos(1)) &= .857553215846 \\ \cos(\cos(\cos(1))) &= .654289790498 \\ \cos(\cos(\cos(\cos(1)))) &= .793480358742 \\ \cos(\cos(\cos(\cos(\cos(1)))))) &= .701368773623\end{aligned}$$

As I continued pressing the cosine button, the numerical results kept oscillating to values above and below, but each time closer to, the final limiting value of approximately .739085133215.

Of course there’s nothing special about the cosine function; any function can be iterated, but not all iterated functions have the same nice convergence properties that the cosine function has. In this paper, we’ll look at various forms of iteration.

## 2 A Simple Practical Example

Suppose you put some money (say  $x$  dollars) in a bank at a fixed interest rate. For example, suppose the bank offers simple interest at 10% per year. At the end of one year, you will have your original  $x$  dollars plus  $(.10)x$  dollars of interest, or  $x + (.10)x = (1.10)x$  dollars. In other words, if you begin with any amount of money, one year later you will have that amount multiplied by 1.10.

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<sup>1</sup>One radian is equal to about 57.2957795131 degrees. If you’ve never seen it, this seems like a strange way to measure angles but it makes very good sense. In a unit circle (a circle with radius 1), the circumference is  $2\pi$ , and if we measure in terms of that length instead of in terms of 360 degrees, we find that  $2\pi$  radians is 360 degrees, from which the conversion above can be derived.

Suppose you'd like to know how much money you have after 5 or 10 years. If you consider the increase in value over one year to be a function named  $f$ , then we will have:

$$f(x) = (1.10)x.$$

The function  $f$  will take any input value and tell you the resulting output value if that input value is left in the bank for one year. Thus if you start with  $x$  dollars, then after one year, you will have  $f(x)$  dollars. At the beginning of the second year you have  $f(x)$  dollars, so at the end of the second year, you'll have  $f(f(x))$  dollars. Similar reasoning yields  $f(f(f(x)))$  dollars after the third year,  $f(f(f(f(x))))$  dollars at the end of the fourth year, and so on.

It would be nice to have an easier notation for function iteration, especially if we iterate 100 or 1000 times. Some people use an exponent, like this:

$$f(f(f(f(x)))) = f^4(x),$$

but there's a chance that this could be confused with regular exponentiation (and it could, especially with functions like the cosine function, mentioned above). So we will use the following notation with parentheses around the iteration number:

$$f(f(f(f(x)))) = f^{(4)}(x).$$

As with standard exponentiation, we'll find it is sometimes useful to define:

$$f^{(0)}(x) = x.$$

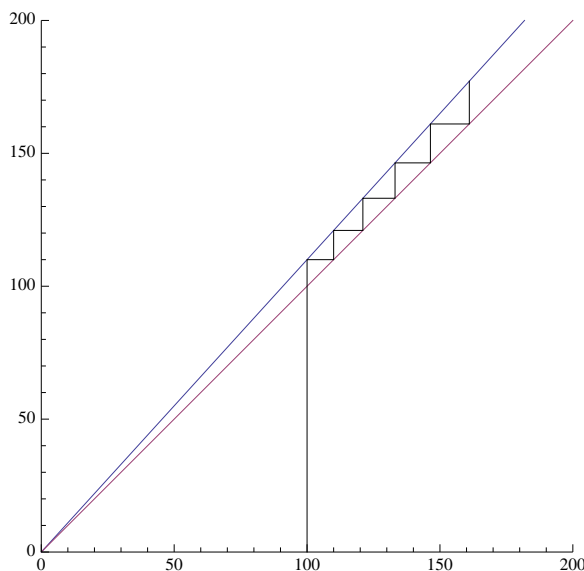
Returning to our example where  $f(x)$  represents the amount of money in your account a year after investing  $x$  dollars, then the amount you'll have after 10 years would be  $f^{(10)}(x)$ .

It is fairly easy to derive exactly the form of  $f^{(n)}(x)$  for this example. Since each year, the amount of money is multiplied by 1.10, we can see that:

$$f^{(n)}(x) = (1.10)^n x,$$

and this even works for the case where  $n = 0$ .

Although this example is so simple that it is easy to give a closed form for  $f^{(n)}(x)$ , we will use that example to show how iteration can be analyzed graphically. First, imagine the graph of  $f(x) = (1.10)x$  versus  $x$ .



In the graph above, two lines are plotted that pass through the origin. The upper line corresponds to the function  $y = f(x) = (1.10)x$  and the lower one, to the function  $y = x$  which forms a  $45^\circ$  angle with the  $x$  and  $y$  axes. Now suppose we begin with an investment of \$100. To find out how much money we have after one year, find the 100 on the  $x$ -axis and go up to the line  $f(x)$ . The height of that line (which will be 110) represents how much we'll have after one year. But now we would like to put that 110 back into the function  $f$ , so we need to find 110 on the  $x$ -axis and look above that.

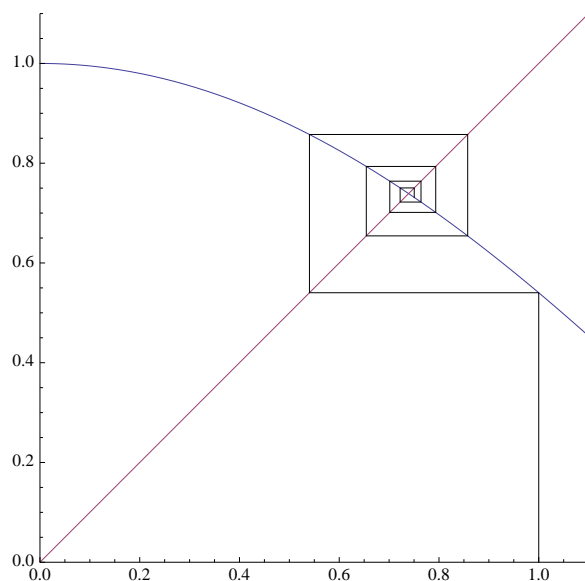
But what this amounts to is copying the  $y$ -value (the height of the line from the  $x$ -axis to the line  $f(x)$ ) to the  $x$ -axis. Here is where the line  $y = x$  suddenly becomes useful: If we begin at the point  $(100, 110)$  and move horizontally to the line  $y = x$ , we will be situated exactly over the value 110 on the  $x$ -axis (since on the line  $y = x$ , the  $y$ -coordinate is the same as the  $x$ -coordinate). Since we're already exactly over 110, we can just go up from there to the line  $f(x)$  to find the value of our investment after two years: \$121.

To find the value after three years, we can do the same thing: the height of the point  $(110, 121)$  needs to be copied to the  $x$ -axis, so move right from that point to  $(121, 121)$ , putting us over 121 on the  $x$ -axis, and from there, we go up to  $f(121)$  to find the amount after three years: \$133.10.

The same process can be used for each year, and the final height of the zig-zagging line at the upper-rightmost point represents the value of the original \$100 investment after six years: about \$177.16.

### 3 General Iteration

A little thought shows us that there is nothing special about the  $f(x)$  we used in the previous section in a graphical interpretation. We can see what happens with the cosine function mentioned in the introduction, by using exactly the same sort of graph, except that  $f(x) = \cos(x)$  and we will begin our iteration with  $x = 1$ :

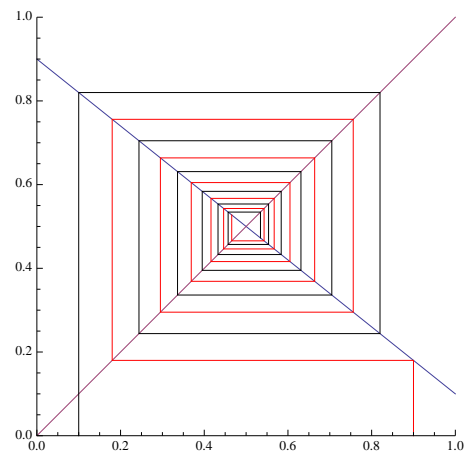
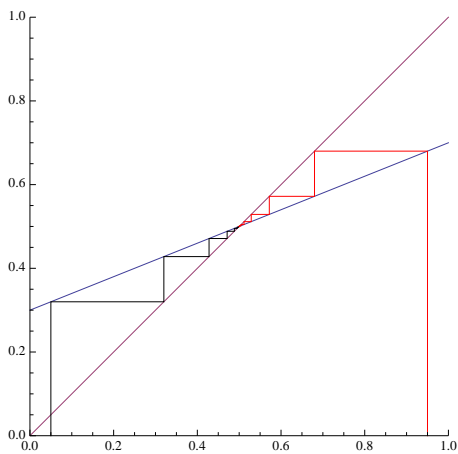
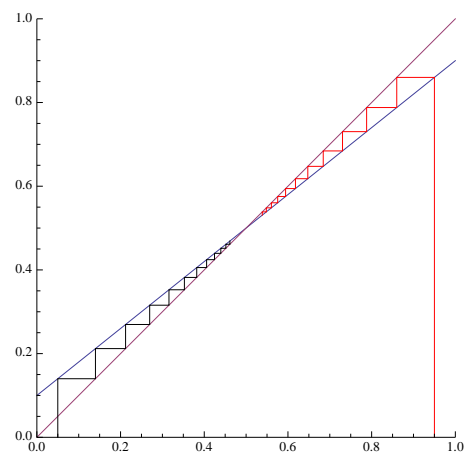
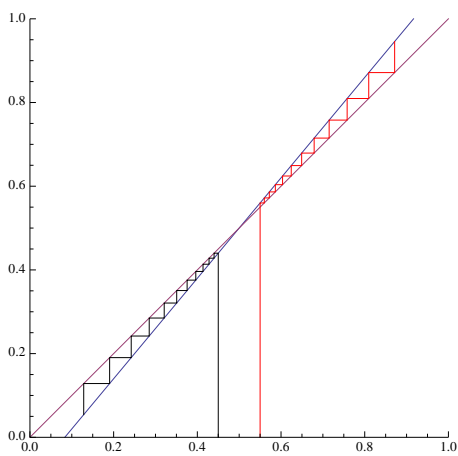


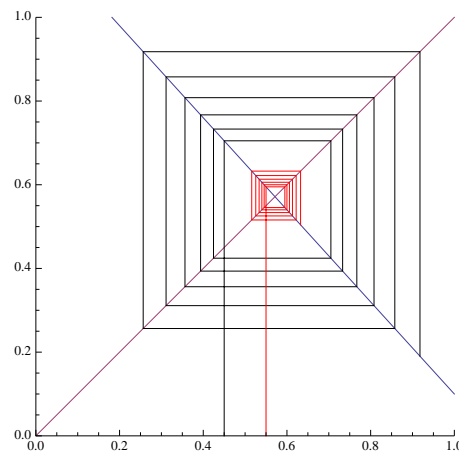
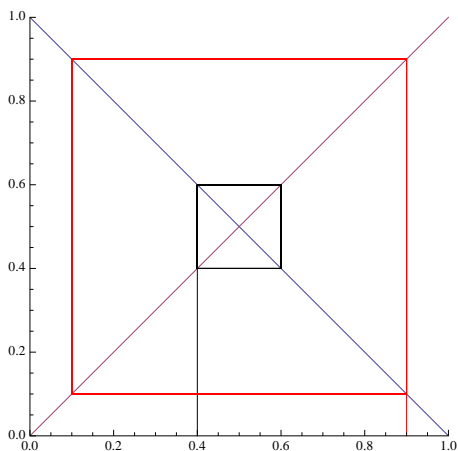
Although the picture looks different (instead of a zig-zagging line, we've got a spiral), exactly the same thing is going on. We begin with an input value of  $x = 1$ , we go up to the line  $y = \cos(x)$  to find the result of one press of the cosine button. The height of that point has to be used as an input, so we move horizontally from it to the line  $y = x$ , at which point we're above a point  $x$  which is equal to the  $y$ -coordinate of the point  $(1, \cos(1))$ . Move vertically from there to the line  $y = \cos(x)$ , then horizontally to  $y = x$ , and repeat as many times as desired. It should be clear from the illustration

that as more and more iterations are performed, the spiral will converge to the point where the line  $y = x$  meets the curve  $y = \cos(x)$ , and that point will have a value of approximately  $(.739085133215, .739085133215)$ .

The major difference between the two examples is that it should be clear that in the case of bank interest, the zig-zag line will keep going to the right and up forever, the spiraling line for the cosine function will converge more and more closely to a limiting point. What we would like to do is examine the geometry of the curves  $f(x)$  that either causes them to converge (like the cosine curve) or to diverge (like the bank interest function). We'd also like to know if there are other possibilities (other than convergence or divergence). We will examine graphically a number of examples in the next section.

## 4 Graphical Examples





Every example on the previous page shows what happens for a particular function of  $x$ , assuming two different starting points. All of the functions on that page are linear (straight lines) and they illustrate the convergence (or divergence) properties. All the examples have the line  $y = f(x)$  crossing the line  $y = x$  at  $(.5, .5)$ . Basically, the only difference between the different functions  $f$  is the slope of that line. We call the point of intersection of  $y = f(x)$  and  $y = x$  a “fixed point”, since if the input happened to have exactly that value, the output would be the same, or fixed.

In the example on the upper left we have basically the same sort of situation we had with the bank account balance except that in the bank balance case, the two lines met at  $(0, 0)$ . In this case, if we start a bit above the intersection, or fixed point, the values diverge, getting larger and larger. If we begin below the intersection, the values diverge, but to smaller and smaller values. The slope of  $f(x)$  in this example is 1.2.

On the upper right, the slope of  $f(x)$  is 0.8, and now the iterations converge to the fixed point from both directions.

The example on the middle left is similar, but the slope is less: only 0.4. The convergence is the same, from above or below, but notice how much faster it converges to the fixed point.

The next three examples show what happens when  $f(x)$  has a negative slope. The example on the middle right, the slope is negative, but not too steep (in fact, it is  $-0.8$ ). Convergence occurs from any starting point, but instead of zig-zagging to the fixed point, it spirals in.

The example on the bottom left shows a very special case where the slope of  $f(x)$  is exactly  $-1$ . In this case, from any starting point there is neither convergence nor divergence; the output values fall into “orbits” of two values.

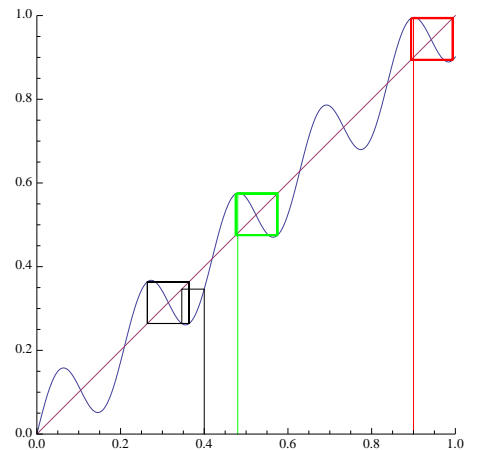
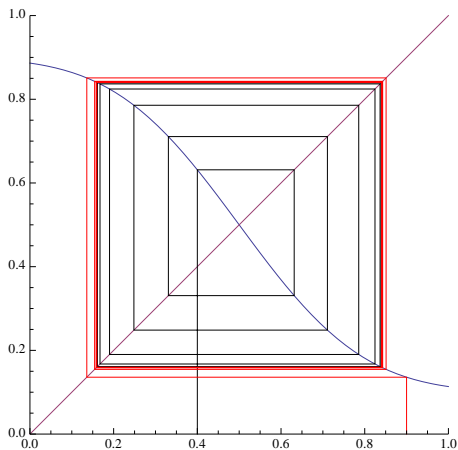
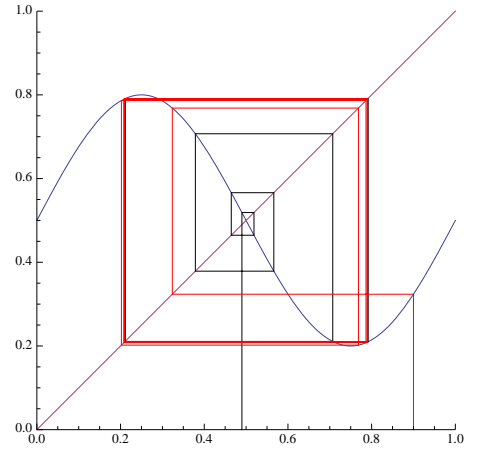
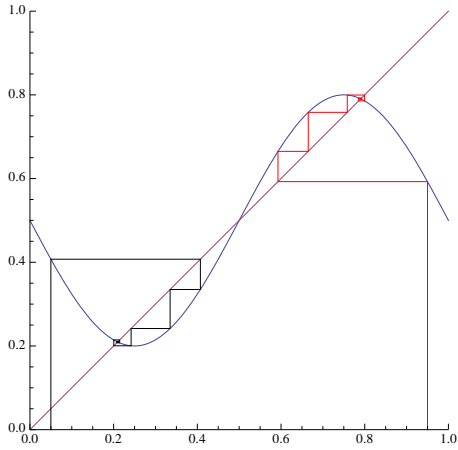
Finally, in the example on the lower right, the slope is negative and steeper than  $-1.0$ , and any input, except for the fixed point itself, will diverge in a spiral pattern, as shown.

The examples on the previous page pretty much cover all the possibilities for linear functions  $f(x)$ . If the slope, in absolute value, is less than 1, iterations converge to the fixed point. If the slope’s absolute value is greater than 1, it diverges, and if the slope’s value is equal to 1 or  $-1$ , there is a fixed orbit of one or two points. (Note that the function with slope 1 is  $y = f(x) = x$  which is identical to the line  $y = x$ , so every point is a fixed point whose orbit contains exactly that point.)

It will turn out that if the function  $f$  is not a straight line, the resulting situation is usually not too different. Compare the example of the cosine function with the similar example on the middle right. In both cases the slopes of the (curve, line) are negative, but between  $-1$  and 0.

In the next section we will examine more complex situations where the iteration results are not quite so simple.

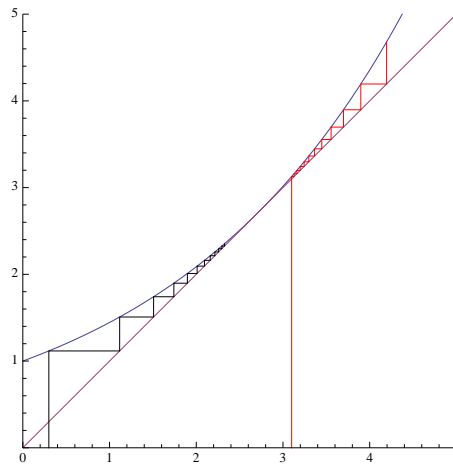
## 5 More Graphical Examples



Here are four more examples. The graph in the upper left shows that there may be more than one fixed point.

The other three illustrate limit cycles. In the examples in the upper right and lower left, different starting points converge not to a point, but to a cycle. On the upper right, the cycles first step inside the limiting cycle and then spiral out. In the lower right, one cycles in while the other cycles out.

Finally, the example in the lower right shows that there may be many limiting cycles, and the particular cycle into which a starting point falls depends on the position of the starting point.



The figure above shows another interesting example. The curve  $f(x)$  is actually tangent to the line  $x = y$ , and initial values less than the point of tangency converge to that point, while initial values larger than it diverge.

You may find it interesting to try to construct examples with combinations of fixed points, cycles and sequences that diverge.

## 6 A Simple Practical Example

Suppose you want to approximate the square root of a number, for example  $\sqrt{3}$ . The following strategy works:

Make a guess, say  $x$ . If  $x$  is the correct answer, then  $x^2 = 3$ , and another way of saying that is:  $x = 3/x$ . If  $x$  is not the exact square root, then if  $x$  is too small,  $3/x$  will be too large and vice-versa. Thus the two numbers  $x$  and  $3/x$  will lie on both sides of the true square root, and thus their average is likely to be closer to the true square root than either one of them, but in any case, the true root can be limited to a smaller interval.

If  $x$  is not the exact square root, then the value  $x_1 = (x + 3/x)/2$  is closer to the square root than  $x$  was. We can then use  $x_1$  as a new guess at the square root and repeat the procedure above as many times as desired to obtain more and more accurate approximations. The following illustrates this method to find  $\sqrt{3}$ , starting with an initial guess  $x_0 = 1$ , and approximated to 20 decimal places. The final line displays the actual result, again accurate to 20 decimal places:

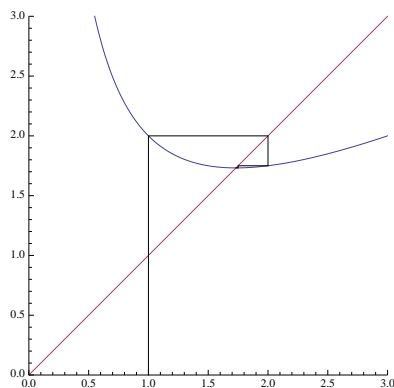
$$\begin{aligned}
 x_0 &= 1.00000000000000000000 \\
 x_1 = (x_0 + 3/x_0)/2 &= 2.00000000000000000000 \\
 x_2 = (x_1 + 3/x_1)/2 &= 1.75000000000000000000 \\
 x_3 = (x_2 + 3/x_2)/2 &= 1.73214285714285714286 \\
 x_4 = (x_3 + 3/x_3)/2 &= 1.73205081001472754050 \\
 x_5 = (x_4 + 3/x_4)/2 &= 1.73205080756887729525 \\
 \sqrt{3} &= 1.73205080756887729352
 \end{aligned}$$

The fifth approximation is accurate to 17 decimal places. As you can see, this method converges rather rapidly to the desired result. In fact, although we will not prove it here, each iteration approximately doubles the number of decimal places of accuracy.

Obviously, what we are doing is iterating the function  $f(x) = (x + 3/x)/2$  and approaching the fixed point which will be the square root of 3. This is because when  $f(x) = x$  then  $x$  must be the square root of 3. Also, obviously, there is nothing

special about 3. If we wanted to approximate the square root of  $n$ , where  $n$  is fixed, we simply need to iterate the function  $f(x) = (x + n/x)/2$ .

The following figure is a graphical illustration of the convergence of the iterates of  $f(x)$  as described above, beginning with an initial guess of  $x = 1$ . After the first few steps, due to the flatness of the curve, you can see that the convergence is very rapid.



An interesting strategy that can be performed on almost any iterated function is this: we can double the speed of the convergence using a different function. In the example above, we approximated  $\sqrt{3}$  by iterating  $f(x) = (x^2 + 3)/(2x)$ . Suppose we iterate  $f(f(x))$ ? That should go twice as fast. In this case,

$$f(f(x)) = \frac{\left(\frac{x^2+3}{2x}\right)^2 + 3}{2\left(\frac{x^2+3}{2x}\right)}.$$

A little algebra yields:

$$f(f(x)) = \frac{x^4 + 18x^2 + 9}{4(x^3 + 3x)}.$$

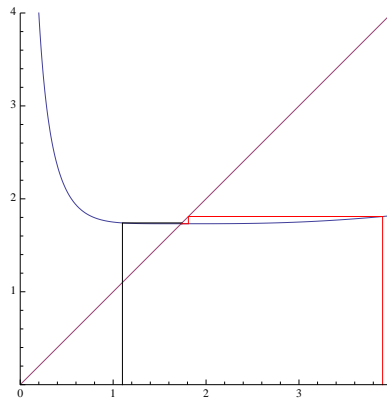
Iterations of that function are shown below, and it's obvious that the new function is a lot flatter, so convergence will be a lot faster.

Obviously, there is no reason we couldn't do the algebra for three, four or more iterations in the same way. The disadvantage, of course, is that the functions that we evaluate are more complicated, so the calculation time for each step may increase more rapidly than the time gained.

Here are three iterations to 30 decimal places of the function  $f(f(x))$  calculated above, beginning with a (bad) initial guess of 4:

$$\begin{aligned} f(4) &= 1.81907894736842105263157894737 \\ f(f(4)) &= 1.73205205714701213020972696708 \\ f(f(f(4))) &= 1.73205080756887729352744640016 \\ \sqrt{3} &= 1.73205080756887729352744634151 \end{aligned}$$





## 7 A Simple Impractical Example

The following is not a particularly good way to do it, but it does use a simple iterated function to converge to the value of  $\pi$ . Unfortunately, to do the calculations requires that we be able to calculate the sine function to many decimal places which is at least equally difficult as computing  $\pi$  using standard methods.

But here's the function to iterate:  $p(x) = x + \sin(x)$ . It's a good exercise to plot the function and see why a fixed point is at  $x = \pi$ . In any case, here are the results of repeatedly applying that function from  $x_0 = 1$  (accurate to 50 decimal places):

```

x_0 = 1.0000000000000000000000000000000000000000000000000000000000000000000000
x_1 = 1.8414709848078965066525023216302989996225630607984
x_2 = 2.8050617093497299136094750235092172055890412289217
x_3 = 3.1352763328997160003524035699574829475623914958979
x_4 = 3.1415926115906531496011613183596168589038475058108
x_5 = 3.1415926535897932384626310360379289800865645064638
x_6 = 3.1415926535897932384626433832795028841971693993751
  pi = 3.1415926535897932384626433832795028841971693993751

```

The reason this works is because when  $p(x) = x$  we will have  $\sin(x) = 0$  which occurs when  $x = \pi$ . Of course different starting points for the iteration can converge to other places where  $\sin(x) = 0$ . If, for example, you start with an initial guess of 7, the iterations will converge to  $3\pi$ :

```

x_0 = 7.0000000000000000000000000000000000000000000000000000000000000000000000
x_1 = 7.65698659871878909039699909159363517793687001049
x_2 = 8.63764574509069809866965497690697020299422915766
x_3 = 9.34597762267712372479785467662073450965918297827
x_4 = 9.42469643439035213789486286303710694430607020889
x_5 = 9.4247796076928940385571979323074280697295349770

```

$$\begin{aligned}
x_6 &= 9.42477796076937971538793014983850865259138543205 \\
x_7 &= 9.42477796076937971538793014983850865259150819813 \\
3\pi &= 9.42477796076937971538793014983850865259150819813
\end{aligned}$$

## 8 Fixed Point Iteration Method

We have a method that sometimes works well to find roots of equations of the form  $f(x) = x$ . Usually we are searching for roots of equations that have the form  $f(x) = 0$ , but many times those equations can be modified so that the roots are fixed points of other iterated functions.

As a first example, let's try to find a root of the polynomial equation:

$$x^5 - x - 10 = 0.$$

We want to convert this to have the form  $g(x) = x$ , and the most straightforward method would be to let  $g(x) = x^5 - 10$ . Unfortunately, almost any starting guess blows up rapidly. Here we start at 1.0:

$$\begin{aligned}
x_0 &= 1.00000 \\
x_1 &= -9.00000 \\
x_2 &= -59059.0 \\
x_3 &= -7.1850607 \times 10^{23}
\end{aligned}$$

But if we convert the original equation to this form:

$$(x + 10)^{1/5} = x,$$

so  $g(x) = (x + 10)^{1/5}$  we obtain the following sequence of approximations to the root:

$$\begin{aligned}
x_0 &= 1.00000000000000000000 \\
x_1 &= 1.61539426620217800151 \\
x_2 &= 1.63307748451253191164 \\
x_3 &= 1.63357441979568861679 \\
x_4 &= 1.63358837598049726528 \\
x_5 &= 1.63358876792624865047
\end{aligned}$$

and the convergence continues to yield  $x = 1.633588779251719496427$  after 14 iterations.

There are, of course, other ways to convert the problem. For example, the equation is also equivalent to:

$$\left(\frac{x + 10}{x^2}\right)^{1/3} = x.$$

This will also converge to the same root, but to obtain the same accuracy as the previous method requires 38 steps.

You may have to try a few different transformations of the equation you'd like to solve to find a form suitable for this method, but one can be found surprisingly easily. For example, if you want to solve this equation:

$$x^x = 3,$$

after some experiments, you find that it can be converted to this form:

$$e^{(\log(3)/x)} = x$$

and when you iterate it, it slowly converges to about 1.82545502292483 which is close to the actual root.

## 9 Fixed Loops and Sharkovsky's Theorem

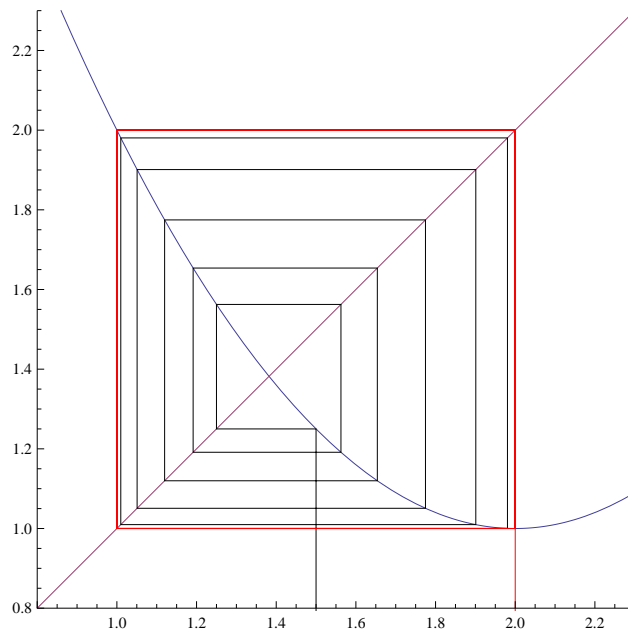
Thanks to Alon Amit, from whom I stole these ideas.

Some of the examples of iteration we have seen lead not to fixed points but to loops. For example, if you iterate the function  $f(x) = -x$  then if your starting point is any non-zero number the iteration will cycle back and forth between the initial number and its negative. At zero, of course,  $f$  has a fixed point:  $f(0) = 0$ .

Here's another example of a function that has a 2-cycle:

$$f(x) = x^2 - 4x + 5.$$

Note that  $f(1) = 2$  and  $f(2) = 1$ , so if we begin our iterations at either 1 or 2, the function values will alternate between those two values. If we begin at, say, 1.5, the iterated function values will cycle out, approaching the cycle of 1 and 2. In the figure below, the red lines indicate a loop starting at 2. The black spiral shows the iterated values beginning at 1.5.



Note that the function has two (unstable) fixed points at:

$$x = \frac{5 \pm \sqrt{5}}{2},$$

where  $x = x^2 - 4x + 5 = f(x)$ .

In both this example and the initial example of this section, the functions had at least one 2-cycle, and both of them also had a fixed point. In fact, it is not hard to show that any continuous function that has a 2-cycle will also have a fixed point, and in fact, a fixed point between the two values that form the 2-cycle). Suppose that  $f$  is a continuous function with  $f(a) = b$  and  $f(b) = a$ , where  $a$  and  $b$  are different. We might as well assume that  $a < b$ .

It's easy to have a geometric intuition about why there must be a fixed point. The points  $(a, b)$  and  $(b, a)$  are on the graph of  $f$ , but  $(a, b)$  is above the line  $x = y$  and  $(b, a)$  is below that line. Since  $f$  is continuous and connects two points on opposite sides of the line, it must cross the line.

For a more formal proof, consider the function  $g(x) = f(x) - x$ .  $g(a) = b - a$  is positive and  $g(b) = a - b$  is negative. Since  $g$  is continuous (it's the difference of two continuous functions), the intermediate value theorem from calculus tells us that there must be some  $c$  with  $a < c < b$  such that  $g(c) = 0$ . (Intuitively,  $g$  starts above zero and moves continuously to a point below zero, so it must hit zero somewhere in between.) But  $g(c) = f(c) - c = 0$ , so  $f(c) = c$  and so  $c$  is a fixed point of  $f$ .

## 9.1 What About 3-Cycles?

Is it possible to have a function that, when iterated, has loops of length 3? In other words, there are three different numbers,  $a$ ,  $b$ , and  $c$  such that:

$$f(a) = b; f(b) = c; f(c) = a.$$

The answer is yes, and here is a great example<sup>2</sup>:

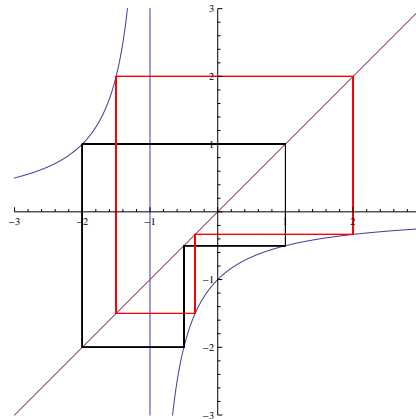
$$f(x) = -\frac{1}{1+x}.$$

Try starting with  $x = 1$ .  $f(1) = -1/2$ , then  $f(-1/2) = -2$ , and  $f(-2) = 1$ .

Try starting with  $x = \pi$ :  $f(\pi) = -1/(1 + \pi)$ , then (skipping the algebra)  $f(-1/(1 + \pi)) = -(1 + \pi)/\pi$ , and finally  $f(-(1 + \pi)/\pi) = \pi$ , so  $\pi$  is also a member of a 3-cycle. What's interesting about this is that if we had calculated with an arbitrary number  $x$  instead of  $\pi$ , the algebra would be identical, and this shows that (almost) every number is in its own 3-cycle. The "almost" is because, for example, if  $x = -1$  we would have division by zero, and if  $x = 0$  then  $f(f(0)) = f(-1)$  which also would cause division by zero. But any real value of  $x$  other than 0 or  $-1$ , on iteration, will fall into a 3-cycle<sup>3</sup>.

Note that the function  $f(x)$  is not continuous: it blows up at  $x = -1$ . Also note that the three values in the loop lie on both sides of  $-1$ . In fact, it will be impossible to find a function that consists only of 3-cycles that does not behave this way (being discontinuous and having the values on both sides of the discontinuity).

Here is a graph of  $f(x)$  and of  $x = y$ , showing the two 3-cycles generated starting with the values 1 and 2:



<sup>2</sup>Another function that behaves in almost exactly the same way is  $f(x) = 1 - 1/x$ .

<sup>3</sup>Note that if we play fast and loose with our mathematics and say that division by zero yields " $\infty$ " and that " $\infty + 1 = \infty$ " and that " $1/\infty = 0$ " then the three numbers 0,  $-1$ , and  $\infty$  form a 3-cycle. This can be formalized to make sense if we work with the projective real line (which includes a special "point at infinity").

## 9.2 Larger Cycles

It is easy to construct continuous (even polynomial) functions with at least one cycle of any desired length. As an example, let's construct a function  $f(x)$  such that  $f(0) = 1$ ,  $f(1) = -1$ , and  $f(-1) = 0$ . The trick we'll use depends on the fact that any (non-collinear) set of points in the plane determines a unique quadratic function of the form:  $f(x) = ax^2 + bx + c$ . In this case, we are trying to find values of the unknowns  $a$ ,  $b$ , and  $c$  so that the function behaves as desired.

If we simply substitute the values of 0, 1, and  $-1$  into  $f$  we obtain:

$$\begin{aligned}0a + 0b + c &= 1 \\1a + 1b + c &= -1 \\1a - 1b + c &= 0\end{aligned}$$

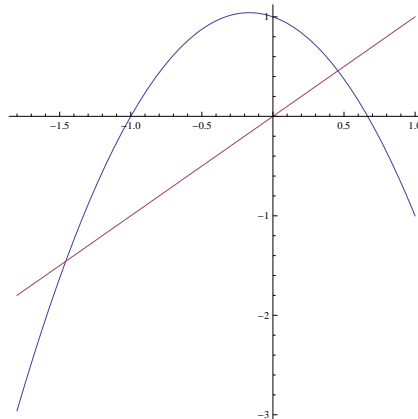
Here we have three equations and three unknowns and it is quite easy to solve, especially because the first equation directly tells us that  $c = 1$ . If we add the other two equations together and substitute 1 for  $c$  we obtain:

$$2a + 2 = -1,$$

so  $a = -3/2$  and knowing  $a$  and  $c$  we can easily solve for  $b$ :  $b = -1/2$ . The desired function is:

$$f(x) = -\frac{3}{2}x^2 - \frac{1}{2}x + 1.$$

It's easy to check our work: just substitute 0, 1, and  $-1$  into the equation and see that we obtain 1,  $-1$ , and 0, respectively. In fact, the function also has two fixed points that are easily calculated if you solve  $f(x) = x$ . The values of those fixed points are  $x = (-3 + \sqrt{33})/6$  and  $x = (-3 - \sqrt{33})/6$ . Here is a graph, including the line  $x = y$ , and the fixed points are where the parabola intersects that line. Note that the units are indicated by different lengths on the  $x$  and  $y$  axes so the line  $x = y$  is not the usual  $45^\circ$  line:



There is nothing special about finding a polynomial that has a cycle length of 3. If we have four values we would like to cycle, say:  $f(0) = 1$ ,  $f(1) = 2$ ,  $f(2) = 3$ , and  $f(3) = 0$  we can do exactly the same as before except this time we will need a cubic equation of the form  $f(x) = ax^3 + bx^2 + cx + d$  to fit the points. If we substitute the values 0, 1, 2, and 3 into the equation we will have four equations and four unknowns that can be solved, and we will have the desired function. For cycles of length  $n$  we will need a polynomial of degree  $n - 1$  with  $n$  coefficients to be determined.

Our example  $f$  above that contains the 3-cycle has two fixed points. Does it also contain a 2-cycle? If there is a 2-cycle, it will satisfy the following equation:

$$f(f(x)) = x.$$

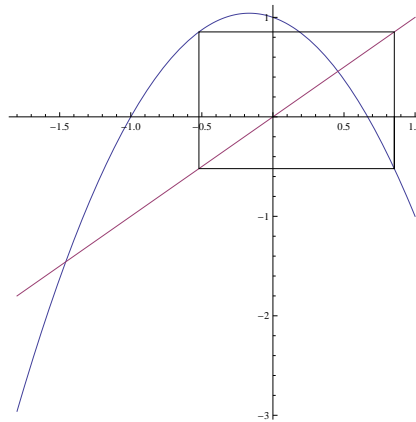
We can calculate  $f^{(2)}(x) = f(f(x))$  by taking the polynomial for  $f(x)$  and substituting it for every  $x$  in the polynomial for  $f(x)$ . Crude, but effective:

$$f^{(2)}(x) = f(f(x)) = -\frac{3}{2}\left(-\frac{3}{2}x^2 - \frac{1}{2}x + 1\right)^2 - \frac{1}{2}\left(-\frac{3}{2}x^2 - \frac{1}{2}x + 1\right) + 1.$$

When we expand  $f^{(2)}(x) = f(f(x))$  we obtain:

$$f^{(2)}(x) = f(f(x)) = -\frac{27}{8}x^4 - \frac{9}{4}x^3 + \frac{39}{8}x^2 + \frac{7}{4}x - 1.$$

If we set this equal to  $x$  and solve it, there are four roots. Two of those roots have to be the fixed points of  $f$  itself so we can factor those out and solve the resulting quadratic. Skipping a lot of straightforward but ugly algebra, we obtain the  $x$ -values for the 2-cycle:  $x = (1 - \sqrt{17})/6$  and  $x = (1 + \sqrt{17})/6$ . Here is a plot showing the 2-cycle:



Since these are the only solutions to the equation, there is only one 2-cycle.

### 9.3 Sharkovsky's Theorem

In our simple polynomial example with a 3-cycle we discovered that it also had a 2-cycle as well as (two) fixed points. Not every function with a 3-cycle has a 2-cycle: in our example  $f(x) = -1/(1+x)$  all but two points were members of unique 3-cycles and the two points did not cycle to each other nor were they fixed points.

But it turns out that if the three points that form a 3-cycle under a function that is continuous on an interval containing those three points then the function will also have at least one 2-cycle and at least one fixed point (which is basically a 1-cycle).

That may sound amazing, but the actual result (Sharkovsky's Theorem, which we will not prove here) is *far, far* more amazing! Sharkovsky's Theorem tells us that if a function has a 3-cycle of that form, then it contains cycles of all possible lengths. Not just a 2-cycle and a 1-cycle, but also a 4-cycle, a 5-cycle, a 1001-cycle, and a  $10^{100}$ -cycle.

The theorem says even more. Consider the following arrangement of all the positive integers:

$$\begin{array}{cccccccc}
 3 & 5 & 7 & 9 & 11 & \dots & (2n+1) \cdot 2^0 & \dots \\
 3 \cdot 2 & 5 \cdot 2 & 7 \cdot 2 & 9 \cdot 2 & 11 \cdot 2 & \dots & (2n+1) \cdot 2^1 & \dots \\
 3 \cdot 2^2 & 5 \cdot 2^2 & 7 \cdot 2^2 & 9 \cdot 2^2 & 11 \cdot 2^2 & \dots & (2n+1) \cdot 2^2 & \dots \\
 3 \cdot 2^3 & 5 \cdot 2^3 & 7 \cdot 2^3 & 9 \cdot 2^3 & 11 \cdot 2^3 & \dots & (2n+1) \cdot 2^3 & \dots \\
 & & & \vdots & & & & \\
 \dots & 2^n & \dots & 2^4 & 2^3 & 2^2 & 2 & 1
 \end{array}$$

The top row contains all the odd numbers starting with 3, the second, all the odd numbers multiplied by 2, the third, all the odd numbers multiplied by 4, and so on, forever. After “forever” we have a list of decreasing powers of 2 down to 1 at the end. Convince yourself that every positive integer appears in this ordering in exactly one place.

Sharkovsky’s Theorem states that if a function makes a cycle of length  $n$  where all the points in the cycle are contained in an interval and the function is continuous on that interval, then if you find  $n$  in the table above, there will be a cycle of length  $m$  for every  $m$  to the right of  $n$ , and for every  $m$  in every row below the row where  $n$  appears. For example, if we have a such a function with a 5-cycle, then we are guaranteed all other cycles of all other lengths except, possibly, a 3-cycle.

It is not particularly easy to find those cycles, however. One method might be this: Suppose we are looking for a 4-cycle of the quadratic function we examined in the previous subsection. If  $x$  is a member of that 4-cycle, then:

$$f(f(f(f(x)))) = f^{(4)}(x) = x.$$

In the same way that we substituted  $f$  into itself once to find the 2-cycle, we can substitute  $f$  into itself three times to make a giant polynomial in  $x$  of degree 16, then set that equal to  $x$  and solve the resulting problem. We would have some help since we know four of the solutions already: the two fixed points and the members of the 2-cycle, but still, we are left with a polynomial of degree 12. At least four of those solutions will be members of the (at least one) 4-cycle.

It turns out that there are only four real roots to this equation, which are the four numbers in the cycle. They are, to about 20 decimal places:

$$\begin{aligned} x_1 &= -0.70084124343476875850 \\ x_2 &= 0.12831861284892250355 \\ x_3 &= 0.61365294896859517409 \\ x_4 &= 0.91114219397033125909 \end{aligned}$$

To find a 5-cycle, we’d be faced with a degree-32 polynomial, and we only know two roots: the two fixed points, et cetera.

Notice that no matter what degree cycles we search for, say  $n$ -cycles, the members of the cycles must be roots of a polynomial equation  $f^{(n)}(x) = x$ . For this  $f$ , all the coefficients will be rational numbers, so all the roots must be algebraic numbers. That means that no transcendental numbers<sup>4</sup> can ever be involved in a cycle. That means that if we begin our iteration with a transcendental number, we can iterate forever and never return to the starting value: the iterations will wander around chaotically.

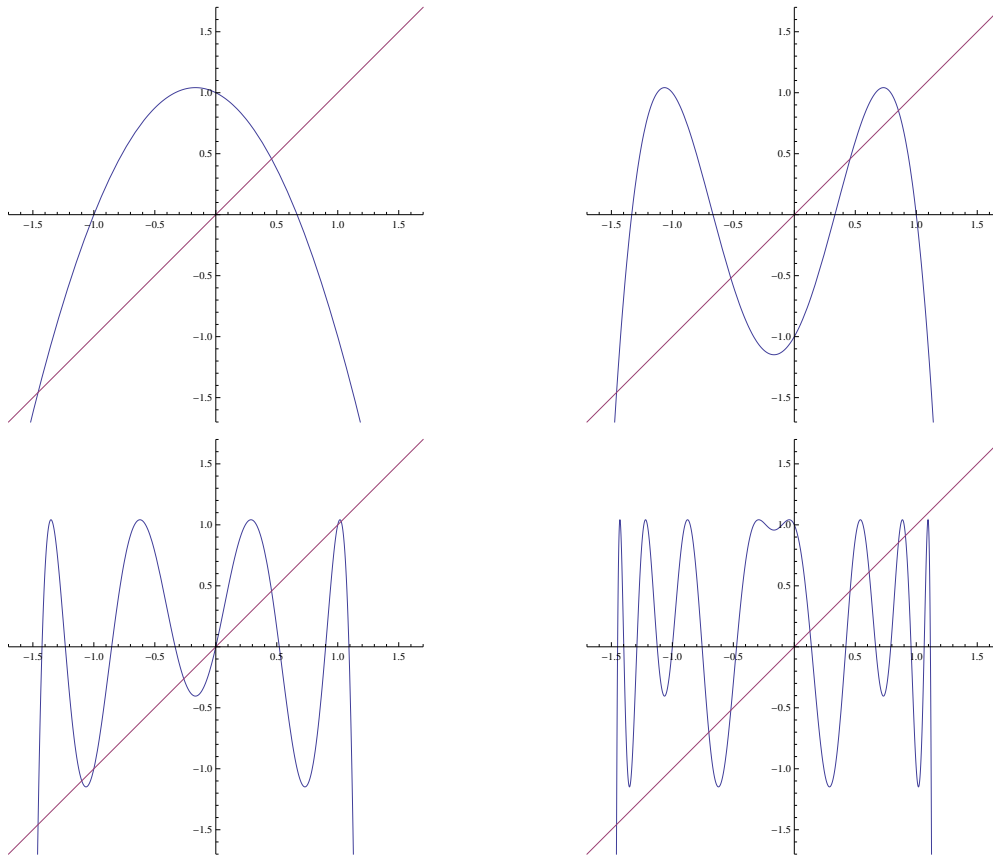
Note that Sharkovsky’s Theorem applies to *any* continuous function, not just polynomials. If you want to construct functions with various different types of cycles that you can specify, there’s no need to stick to polynomials; piecewise linear functions are easy to work with, and very easy to modify to make all sorts of cycles possible.

## 9.4 Interesting Graphs

Just for fun, following are the graphs of  $f(x)$ ,  $f(f(x))$ ,  $f(f(f(x)))$ , and  $f(f(f(f(x))))$ , where  $f(x) = -3x^2/2 - x/2 + 1$ . The intersections of the line  $x = y$  with each occur at the fixed points.

---

<sup>4</sup>A transcendental number is a number that is not the root of any polynomial equation with rational coefficients. Almost all real numbers are transcendental.



Notice that for all four of them, we have the same two fixed points that occur in the first. This makes sense, because if a number is fixed after one iteration, it will be fixed after any number of iterations.

In the second plot, there are two additional fixed points, meaning that  $f(f(x))$  is fixed, or in other words, that those two points lie in a cycle of length 2.

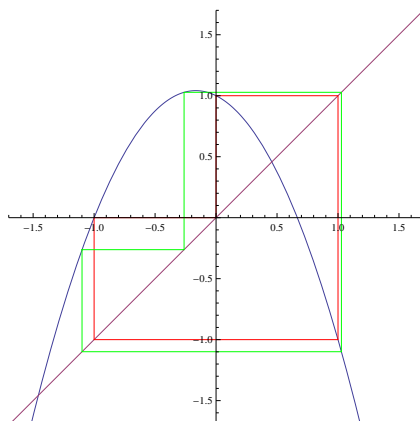
There is a slight surprise in the third one: in addition to the two fixed points, we can see that there are six other fixed points, meaning that there is an additional 3-cycle other than the one we constructed with  $-1$ ,  $0$ , and  $1$ . We can find the values of those additional points by solving the equation  $f(f(f(x))) = x$  which is a polynomial of degree 8. It is not as hard as it seems, though: we know five of the roots, so we really only need to solve a cubic equation. Here are numerical approximations for the three roots:

$$\begin{aligned} x_1 &= -1.0990235147877582363 \\ x_2 &= -0.262267271690776519 \\ x_3 &= 1.0279574531452025548 \end{aligned}$$

It is easy to check with a calculator that  $f(x_1) = x_2$ ,  $f(x_2) = x_3$ , and  $f(x_3) = x_1$  is approximately true.

Here is a plot showing both of these 3-cycles on the same graph:

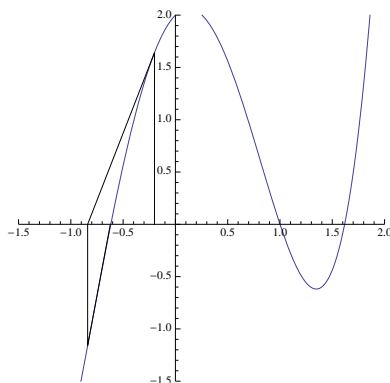




In the fourth graph there are eight intersections: the two fixed points of  $f$ , the two fixed points of  $f(f(x))$  (since iterating  $f(f(x))$  twice is the same as iterating  $f$  four times). And the other four roots are the four we listed in the previous section.

## 10 Newton's Method

The method of approximating a square root used in Section 6 is just a special case of a general method for finding the zeroes of functions.



Newton's method works as follows. (See the figure above.) Suppose that the curved line represents a function for which you would like to find the roots. In other words, you want to find where the function crosses the  $x$ -axis. First make a guess. In the figure, this initial guess is  $x = -0.2$ . Find the point on the curve corresponding to that guess (which will be  $(x, f(x))$ ) and find the tangent line to the curve at that point. That tangent line will intersect the  $x$ -axis at some point, and since the tangent line is a rough approximation of the curve, it will probably intersect near where the curve does. In the example above, this will be at approximately  $x = -0.84$ .

We then iterate the process, going from that intersection point to the curve, finding the tangent to the curve at that point, and following it to where it intersects the  $x$ -axis again. In the figure above, the tangent line and the curve are so similar that it's almost impossible to see a difference between the curve and the line, so the intersection of that line and the  $x$ -axis is almost exactly the root of the function. But if not, the process can be iterated as many times as desired.

In the example above, we are trying to solve  $f(x) = x^4 - 4x^2 + x + 2 = 0$ . The initial guess is  $x_0 = -0.2$ , the second is  $x_1 = -0.839252$ , and the third is  $x_2 = -0.622312$ . To show that  $x_2$  is very close to a root, we have:  $f(-0.622312) =$

-0.0214204.

To find the equation of the tangent line requires a bit of calculus, but assume for a moment that we can do that. In fact, for well-behaved functions, the number  $f'(x)$  represents the slope of the function  $f(x)$  at the point  $(x, f(x))$ .

If a line passes through the point  $(x_0, y_0)$  and has slope  $m$  at that point, the equation for that line is:

$$y - y_0 = m(x - x_0).$$

This is usually called the “point-slope” equation of a line.

In our case, the initial guess will be  $x_0$ ,  $y_0$  will be  $f(x_0)$  and  $m = f'(x_0)$ . (Remember that  $f'(x_0)$  is the slope of the function at  $(x_0, f(x_0))$ .) The equation for the tangent line will thus be:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

This may look confusing, but there are only two variables above,  $x$  and  $y$ . All the others:  $x_0$ ,  $f(x_0)$  and  $f'(x_0)$  are just numbers.

We want to find where this line crosses the  $x$ -axis, so set  $y = 0$  and solve for  $x$ . We obtain:

$$x = x_0 - \frac{f(x_0)}{f'(x_0)},$$

and this equation is used to obtain successive approximations using Newton’s method.

But as before, all we are doing to find better and better approximations for the root of a function is to iterate a particular function. The solution  $x$  in the equation above serves as the next guess for the root of  $f(x) = 0$ .

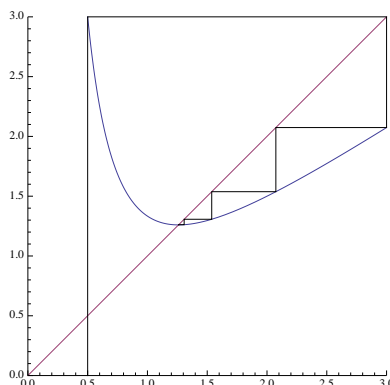
Let us show the graphical interpretation of the iterations performed to use Newton’s method to find the cube root of a number; say, the cube root of 2.

In this case, we would like to solve the equation  $f(x) = x^3 - 2 = 0$ . If you don’t know any calculus, take it on faith that the slope function (called the “derivative” in calculus) is  $f'(x) = 3x^2$ .

If we start with any guess  $x_0$ , the next guess will be given by:

$$x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - 2}{3x_0^2} = \frac{2x_0^3 + 2}{3x_0^2}.$$

The figure at the top of the next page represents the iteration of the function above (beginning with a particularly terrible first guess of  $x_0 = .5$ ) that approaches the cube root of 2, which is approximately 1.25992105.



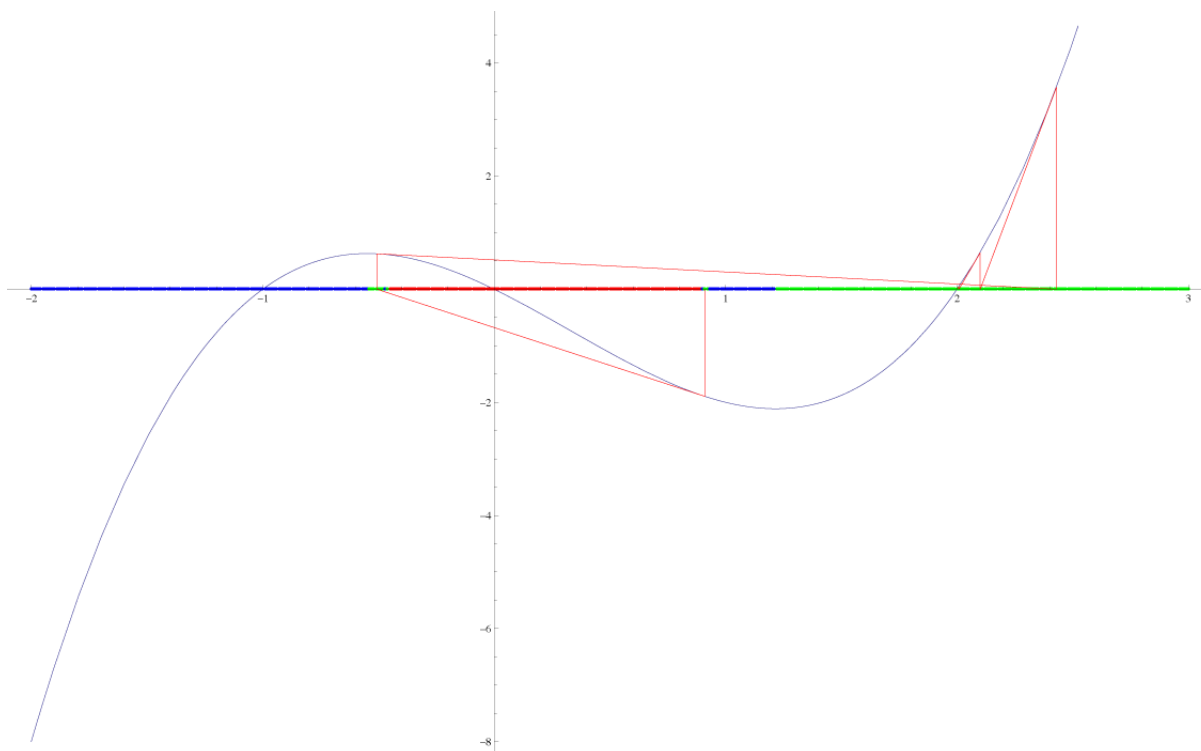
## 10.1 Domains of Attraction

When you use Newton's method to approximate the roots of a polynomial which has more than one root, the particular root to which the method converges depends on your choice of starting point. Generally, if you pick a first guess that is quite close to a root, Newton's method will converge to that root, but an interesting question is this: "Exactly which points on the line, if used as the initial guess, will converge to each root?"

Some points may not converge at all. For example, if your initial guess happened to lie exactly under a local maximum or minimum of the curve, the approximating tangent line would be parallel to the  $x$ -axis, and the iteration process would fail, since that line never intersects the  $x$ -axis. And there are other points where the process might fail. Suppose your initial guess produced a tangent line that intersected the  $x$  axis exactly under a local maximum or minimum. Or if the same thing happened after two iterations, et cetera?

And what happens if you pick a guess near where the function has a local maximum or minimum? The tangent line could intersect far from that initial guess, and in fact, into a region that converged to a root far away from the initial guess.

For any particular root, let us define the "domain of attraction" to be the set of initial guesses on the  $x$ -axis that will eventually converge to that root. As an illustration of some domains of attraction, consider the equation  $f(x) = x(x + 1)(x - 2) = 0$  which obviously has the three roots  $x = -1, x = 0$  and  $x = 2$ . We can still apply Newton's method to initial guesses, and most of them will converge to one of these three roots. In the figure below, points on the  $x$ -axis have been colored blue, red or green, depending on whether they would converge to the roots  $-1, 0$  or  $2$ , respectively.

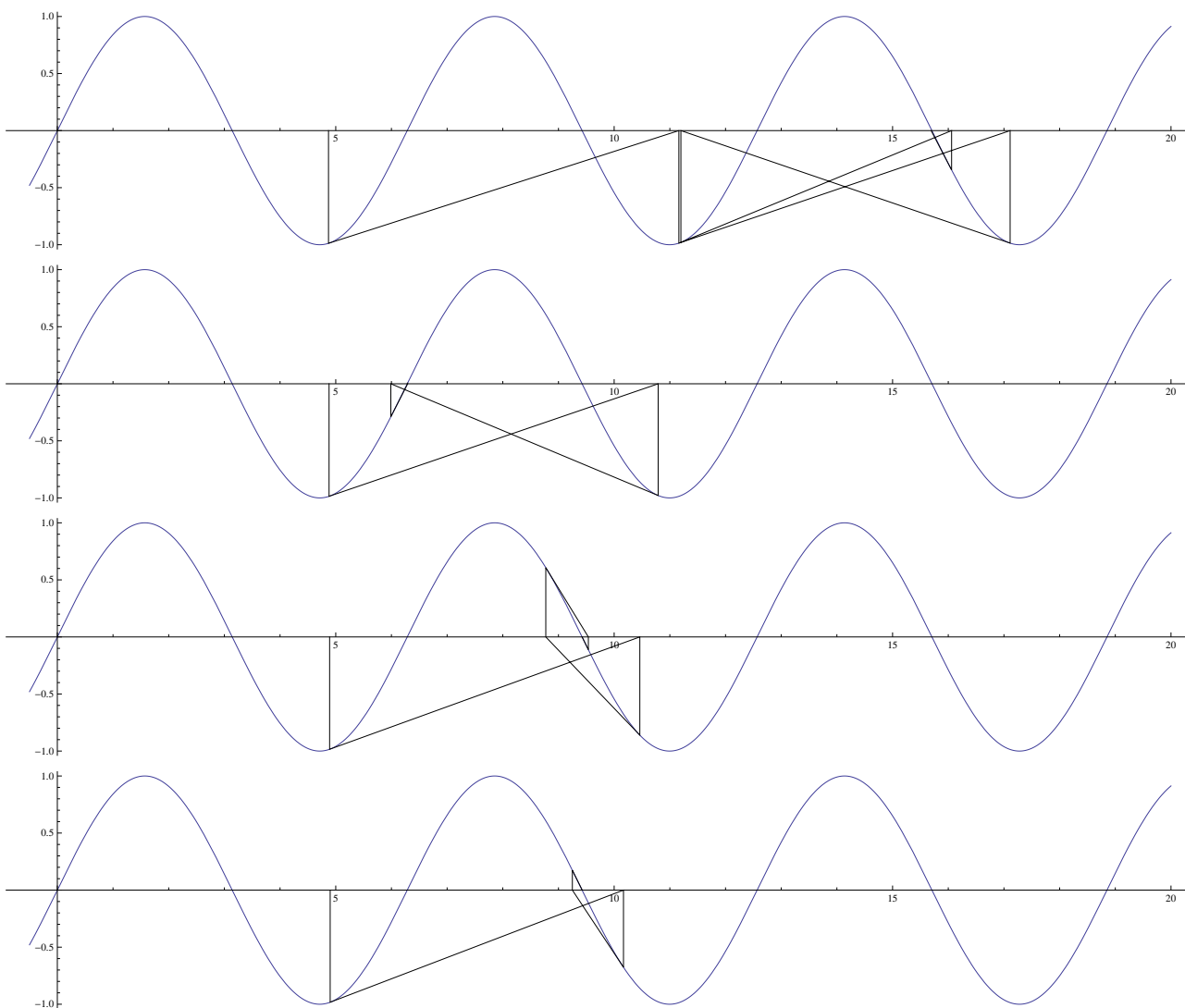


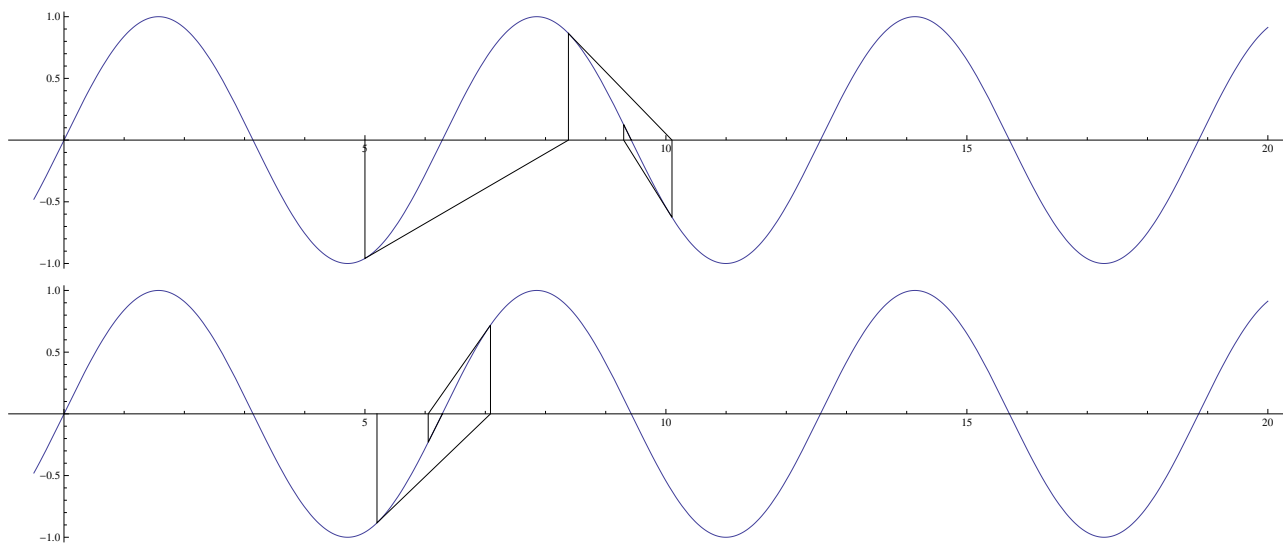
Notice that near the roots, the points are colored solidly, but as we move away from the roots to the regions near where the relative maxima and minima occur, the tangent lines will throw the next guess into regions that converge to non-nearby roots. An example of that is illustrated above. With a particular initial guess that is a bit less than 1.0, successive applications of Newton's method are illustrated by the red line which shows that although the initial guess is a bit closer to the root at

zero, the first approximation takes it to a position closer to the root at  $-1$ , and then out beyond 2, after which it converges rapidly to the root at 2. The image is not infinitely detailed, and in fact, the actual shape of these domains of attraction can be quite complex. In fact, in Section 15 we will see some incredibly complex domains of attraction when Newton's method is extended to search for roots in the complex number plane.

## 10.2 Dependence on starting point

Here is a series of plots illustrating the use of Newton's method on the function  $\sin(x)$ , using starting values of 4.87, 4.88, 4.89, 4.90, 5.00 and 5.20, in that order. Notice how small changes in the input values can produce wildly-different convergence patterns. In every case below, six steps in the approximation are taken. Some converge so rapidly that only the first three steps or so are visible. Some are quite a bit slower.





## 11 Continued Fractions and Similar

Sometimes you see mathematical objects like these:

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$

$$x = 1 + \frac{1}{1 + \frac{1}{1 + \dots}}$$

What do they mean? The usual interpretation is that each expression can be terminated after one, two, three, four, ... steps, and the terminated forms can be evaluated. If the evaluations tend to a limit, then the infinite expression is interpreted to be that limiting value.

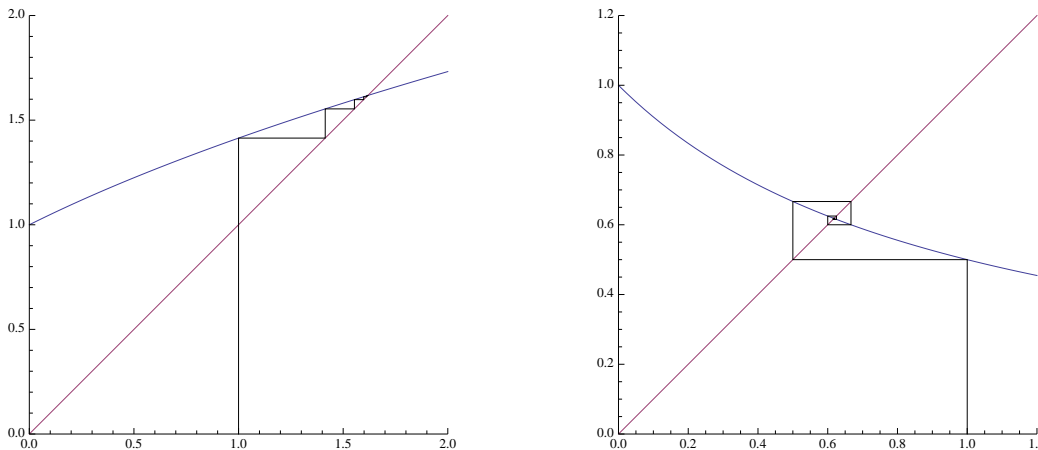
For example, in the first case above, we could evaluate:

$$\begin{aligned} \sqrt{1} &= 1.000000000 \\ \sqrt{1 + \sqrt{1}} &= 1.414213562 \\ \sqrt{1 + \sqrt{1 + \sqrt{1}}} &= 1.553773974 \\ \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} &= 1.598053182 \\ \dots &= \dots \end{aligned}$$

The second case (which is called a “continued fraction”) can be evaluated similarly.

Both, however, can be evaluated using function iteration as well. In the first case, if we have the value  $x_0$  for a certain level of iteration, the next level can be obtained by calculating  $f(x_0)$ , where  $f(x) = \sqrt{1 + x}$ . For the continued fraction, the

corresponding value of  $f(x)$  would be  $f(x) = 1/(1+x)$ . Below are the graphical iterations. On the left is the nested square root and on the right, the continued fraction. Both begin with an initial approximation of 1:



Notice that there is another way to evaluate these expressions, assuming that the limits exist. In the first case, if we let:

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$

then a copy of  $x$  appears inside the expression for  $x$  and we have:

$$x = \sqrt{1 + x}.$$

Squaring both sides, we obtain  $x^2 = 1 + x$ , which has as a root the golden ratio:

$$x = \frac{1 + \sqrt{5}}{2} = 1.61803398874989484820458683437$$

Similarly, if  $x$  is set equal to the continued fraction, we can derive:

$$x = 1/(1 + x),$$

Which becomes  $x^2 + x = 1$ , and has the solution:

$$x = \frac{\sqrt{5} - 1}{2} = .61803398874989484820458683437,$$

one less than the golden ratio.

## 12 Optimal Stopping

Thanks to Kent Morrison, from whom I stole this idea.

Suppose you want to score as high as possible, on average, when you play the following game. The game goes for  $n$  rounds, and you know the value of  $n$  before you start. On each round, a random number uniformly distributed between 0 and 1 is selected. After you see the number, you can decide to end the game with that number as your score, or you can play another round. If play up to the  $n^{\text{th}}$  round, then you get whatever number you get on that last round. What is your optimal strategy?

The topic is called “optimal stopping” because you get to decide when to stop playing.

As with most games, it’s usually a good idea to analyze simple cases first, and the simplest of all is the “game” when  $n = 1$ . It’s not really a game, since once the number is selected, and since there are no more rounds, you will be stuck with that number. Since the numbers are uniformly distributed between 0 and 1, your average score, when  $n = 1$ , is  $1/2$ .

Let us give a name to the expected value of your score for a game with up to  $n$  rounds as  $E_n$ . From the previous paragraph,  $E_1 = 1/2$ .

What about if  $n = 2$ ? What is  $E_2$ ? On the first round a number is chosen, and based on that number, you can decide to use it as your score, or to discard it and play one more round. If your initial score is less than  $1/2$ , it’s clear that you should play again, since on average playing the final round will yield, on average, a score of  $1/2$ . But if your initial score is larger than  $1/2$ , if you discard it, you’ll do worse, on average.

So the optimal strategy for  $n = 2$  is this: Look at the first score. If it’s larger than  $1/2$ , stop with it. If not, discard it, and play the final round. What is your expected score in this game? Half the time you will stop immediately, and since you know your score is above  $1/2$  it will be uniformly picked between  $1/2$  and 1, or in other words, will average  $3/4$ . The other half of the time you will be reduced to playing the game with  $n = 1$ , which you already solved, and your expected score then will be  $1/2$ . So half the time you’ll average  $3/4$  and half the time you’ll average  $1/2$ , yielding an expected value for  $n = 2$  of:

$$E_2 = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{8}.$$

What is  $E_3$ ? After the first round, you have a score. If you discard that score, you will be playing the game with only 2 rounds left and you know that your expected score will be  $5/8$ . Thus it seems obvious that if the first-round score is larger than  $5/8$  you should stick with that, and otherwise, go ahead and play the  $n = 2$  game since, on average, you’ll get a score of  $5/8$ . Thus  $5/8$  of the time you’ll play the game with  $n = 2$  and  $3/8$  of the time you stop, with an average score midway between  $5/8$  and 1, or  $13/16$ . Expected score will be:

$$E_3 = \frac{3}{8} \cdot \frac{13}{16} + \frac{5}{8} \cdot \frac{5}{8} = \frac{89}{128}.$$

The same sort of analysis makes sense at every stage. In the game with up to  $n$  rounds, look at the first round score, and if it’s better than what you’d expect to get in a game with  $n - 1$  rounds, stick with it; otherwise, play the game with  $n - 1$  rounds.

Suppose we have laboriously worked out  $E_1, E_2, E_3, \dots, E_{n-1}$  and we wish to calculate  $E_n$ . If the first score is larger than  $E_{n-1}$ , stick with it, but otherwise, play the game with  $n - 1$  rounds. What will the average score be? Well,  $1 - E_{n-1}$  of the time you’ll get an average score mid-way between  $E_{n-1}$  and 1. The other  $E_{n-1}$  of the time, you’ll get an average score of  $E_{n-1}$ .

The number mid-way between  $E_{n-1}$  and 1 is  $(1 + E_{n-1})/2$ , so the expected value of your score in the game with  $n$  rounds is:

$$E_n = (1 - E_{n-1}) \cdot \left( \frac{1 + E_{n-1}}{2} \right) + E_{n-1} \cdot E_{n-1} = \frac{1 + E_{n-1}^2}{2}.$$

Notice that the form does not really depend on  $n$ . To get the expected score for a game with one more round, the result is just  $(1 + E^2)/2$ , where  $E$  is the expected score for the next smaller game. We can check our work with the numbers we calculated for  $E_1, E_2$  and  $E_3$ . We know that  $E_1 = 1/2$ , so

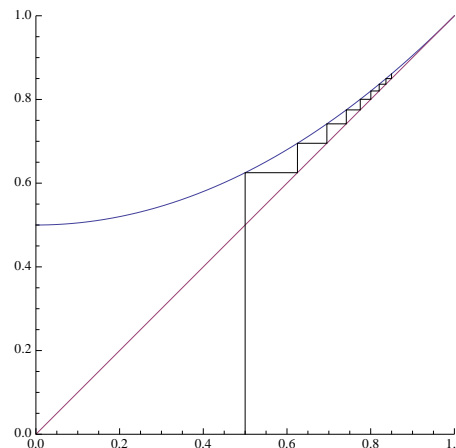
$$E_2 = \frac{1 + (1/2)^2}{2} = \frac{5}{8}, \text{ and } E_3 = \frac{1 + (5/8)^2}{2} = \frac{89}{128},$$

so we seem to have the right formula.

Notice that to obtain the next expected value, we simply take the previous one and plug it into the function  $f(x) = (1 + x^2)/2$ , so basically we just iterate to find successive expected values for successive games with larger  $n$ . Here are the first few values:

$$\begin{aligned}
 E_1 &= \frac{1}{2} = 0.50000 \\
 E_2 &= \frac{5}{8} = 0.62500 \\
 E_3 &= \frac{89}{128} \approx 0.695313 \\
 E_4 &= \frac{24305}{32768} \approx .74173 \\
 E_5 &= \frac{1664474849}{2147483648} \approx .775082 \\
 E_6 &= \frac{7382162541380960705}{9223372036854775808} \approx .800376
 \end{aligned}$$

As before, we can look at the graphical display of the iteration, and it's easy to see from that that the average scores increase gradually up to a limiting value of 1:



### 13 Biology: Heterozygote Advantage

Some knowledge of probability is required to understand this section. A little biology wouldn't hurt, either.

The canonical example of this is the presence of the gene that causes sickle-cell anemia in humans. Basically, if an individual has this disease he/she will not live long enough to have children. Normally, genes like this are eliminated from the population by natural selection, but in human populations in Africa, the gene is maintained in the population at a rate of about 11%. In this section we will see why.

The sickle-cell trait is carried on a single gene, and there are two types:  $A$ , the normal type, does not cause anemia. The sickle-cell gene is called  $a$ . Every individual has two copies of each gene (one from the mother and the other from the father), so there are three possibilities for the genotype of an individual:  $AA$ ,  $Aa$ , or  $aa$ . Suppose that at some point, there is a proportion  $p$  of gene  $A$  in the population and a proportion  $q = 1 - p$  of the gene  $a$ .



We will simply assume that in the next generation that the genes are thoroughly mixed, and therefore, at birth, the three types of individuals will appear with frequencies  $p^2$  (of type  $AA$ ),  $2pq$  (of type  $Aa$ ) and  $q^2$  (of type  $aa$ ).

But years later, when it is time for those individuals to breed, all of the ones of type  $aa$  are dead. In other words, genotype  $aa$  is a recessive lethal gene.

It may also be true that the individuals of types  $AA$  and  $Aa$  have different chances of survival. We don't know exactly what these are, but let us just say that individuals of type  $AA$  are  $(1 + s)$  times as likely to live as individuals of type  $Aa$ . As childbearing adults then, we will find  $(1 + s)p^2$  individuals of type  $AA$  for every  $2pq$  individuals of type  $Aa$ .

We'd like to count the total number of  $a$  genes in the next generation. They can only come from the  $2pq$  proportion having type  $Aa$ , and only half of the genes from those people will be  $a$  since the other half are of type  $A$ . Thus there will be a proportion  $pq$  of them. The total number of genes will be  $2(1 - s)p^2 + 2pq$ .

The proportion of  $a$  genes after breeding will thus be:

$$q' = \frac{pq}{(1 + s)p^2 + 2pq}.$$

But genes are either of type  $A$  or  $a$ , so  $p = 1 - q$  and we have:

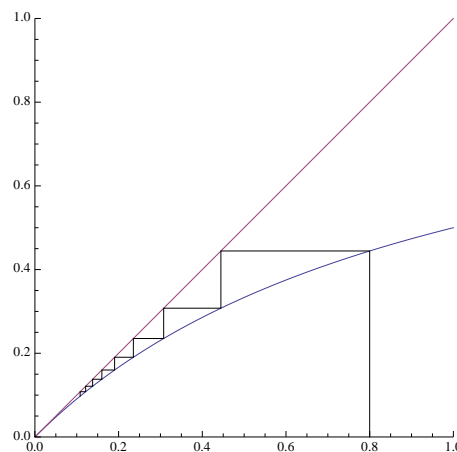
$$q' = \frac{(1 - q)q}{(1 + s)(1 - q)^2 + 2q(1 - q)} = \frac{q}{(1 + s)(1 - q) + 2q}.$$

To obtain the proportion of  $a$  genes in each successive generation, we simply have to put the value of  $q$  into the equation above to find the new value,  $q'$  in the next generation. To find the proportion after ten generations, just iterate 10 times. This is just another iterated function problem!

Let's look at a few situations. Suppose first that the  $AA$  individuals and the  $Aa$  individuals are equally fit. There is no disadvantage to having a single copy of the  $a$  gene. Then  $s = 0$ , and the function we need to iterate looks like this:

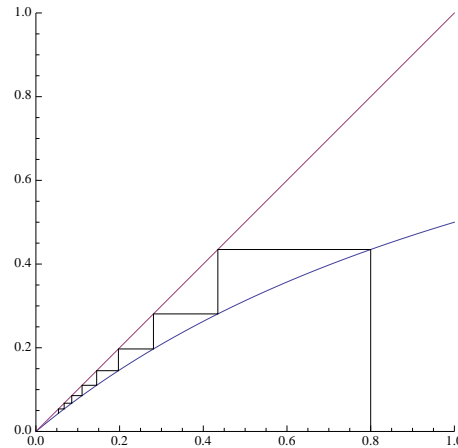
$$q' = f(q) = \frac{q}{1 + q}.$$

Let's assume that for some reason there is a huge proportion of  $a$  genes, say 80%. Here is the graphical iteration in that case:



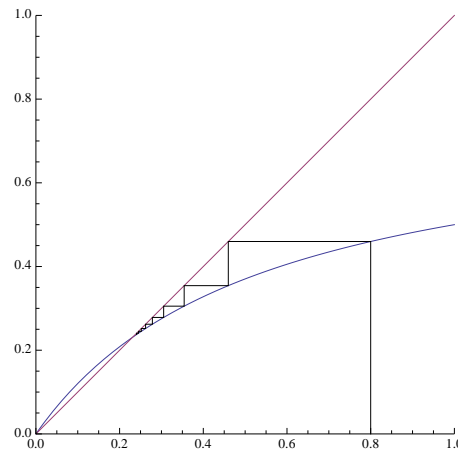
Notice that the fixed point is at zero, so with time, the sickle-cell gene should be eliminated from the population. In other words, the probability that an  $a$  gene will appear drops to zero.

Now suppose that even having a single copy of the  $a$  gene is a disadvantage. Suppose that it is 20% more likely that an  $AA$  individual will live to breeding age than an  $Aa$  individual. This makes  $s = .2$ , and the corresponding graphical iteration looks like this:



Not surprisingly, the sickle-cell gene is again driven out of the population, and if you compare the two graphs, you can see that it will be driven out more rapidly (fewer generations to reduce it the same amount) in this second case. With the same number of iterations, the gene is about half as frequent if the  $AA$  individuals have a 20% advantage over the  $aa$  individuals.

But in the real world, something else happens. In Africa where there is a lot of malaria, individuals with a single copy of the sickle-cell gene (individuals of type  $Aa$ ) actually have an advantage over those of type  $AA$  because they have a better chance of surviving a case of malaria. We can use the same equation, but simply make  $s$  negative. Let's look at the graph with  $s = -.3$ :



Now notice that  $q$  tends to a non-zero limit, in this case, a bit more than 23%. In other words, the advantage to the individuals who have a single copy of the  $a$  gene is enough that the certain deaths of  $aa$  individuals is not enough to eliminate the gene. In fact, a value of  $s = -.3$  is not required; any negative value for  $s$  would make this occur, although a smaller advantage of  $Aa$  would translate to a smaller limiting value of  $q$ .

As in every other case we've looked at, we could find the exact limiting value by setting  $q' = q$ :

$$q' = q = \frac{q}{(1 - .3)(1 - q) + 2q}$$

If we solve that for  $q$ , we obtain:  $q = .23077$ : a bit more than 23%, as we noted above.

Individuals with two copies of the same gene (like  $aa$  or  $AA$ ) are called homozygous, and individuals with one copy of each type are called heterozygotes. In the case of the sickle-cell gene in a malarial area, there is a heterozygote advantage; hence, the title of this section.

## 14 Markov Processes

To understand this section you will need to know something about matrix multiplication and probability.

Imagine a very simple game. You have a board that is nine squares long with a marker beginning on the first square. At each stage, you roll a single die, and advance the marker by the number of steps shown on the die, if that is possible. The game is over when you land on the ninth square. You cannot overshoot the last square, so, for example, if you are on square 5 and roll a 6 the marker stays in place because you cannot advance six spaces. In other words, in order to finish, you need to land on the last square exactly. With this game, we can ask questions like, "How likely is it that the game will have ended after 7 rolls of the die?"

Up to now we have been looking at one-dimensional iteration, but now we will look at a multi-dimensional situation. At every stage of the game, there are exactly 9 possible situations: you can be on square 1, square 2, ..., square 8. At the beginning of the game, before the first roll, you will be on square 1 with probability 1. After the game begins, however, all we can know is the probability of being on various squares.

For example, after one roll there is a  $1/6$  chance of being on squares 2, 3, 4, 5, 6, and 7, and no chance of being on squares 1 or 8, et cetera.

We can, however, easily write down the probability of moving from square  $i$  to square  $j$  on a roll. For example, the chance of moving from square 1 to square 4 is  $1/6$ . The probability of moving from square 4 to itself is  $4/6 = 2/3$ , since only rolls of 1 or 2 will advance the marker. Rolls of 3, 4, 5 or 6 require impossible moves, beyond the end of the board. We can arrange all those probabilities in a matrix that looks like this:

$$M = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The number in row  $i$  and column  $j$  is the probability of moving from square  $i$  to square  $j$  in a single roll. We have put a 1 in row 8, column 8, since if the game is over, it stays over.

As we stated earlier, all we can know is the probability of getting to various squares after a certain number of rolls. At any stage there is a probability of being on square 1, 2, 3, ..., 9. We will write these in a row vector, and that vector, initially (at time zero), looks like this:

$$P_0 = (1, 0, 0, 0, 0, 0, 0, 0, 0).$$

In other words, we are certain to be on square 1.

The nice thing about the matrix formulation is that given a distribution of probabilities  $P$  of being on the 9 squares, if we multiply  $P$  by  $M$  (using matrix multiplication), the result will be a new  $P'$  that shows the odds of being on each square

after a roll of the die. To calculate the probabilities of being on the various squares after 10 rolls, just iterate the matrix multiplication 10 times.

To show how this works, let us call the probability distributions after one roll  $P_1$ , after two rolls  $P_2$ , and so on. Here are the numeric results:

$$\begin{aligned}
 p_1 &= (0, 0.166667, 0.166667, 0.166667, 0.166667, 0.166667, 0.166667, 0, 0) \\
 p_2 &= (0, 0, 0.027778, 0.083333, 0.138889, 0.194444, 0.25, 0.166667, 0.138889) \\
 p_3 &= (0, 0, 0, 0.0185185, 0.0648148, 0.138889, 0.240741, 0.25463, 0.282407) \\
 p_4 &= (0, 0, 0, 0.003086, 0.0246914, 0.0833333, 0.197531, 0.289352, 0.402006) \\
 p_5 &= (0, 0, 0, 0.000514, 0.0087449, 0.0462963, 0.150206, 0.292567, 0.501672) \\
 p_6 &= (0, 0, 0, 0.000086, 0.0030007, 0.0246914, 0.109396, 0.278099, 0.584727) \\
 p_7 &= (0, 0, 0, 0.000014, 0.0010145, 0.0128601, 0.077560, 0.254612, 0.653939)
 \end{aligned}$$

If we look at the last entry in  $p_7$ , we can conclude that after 7 rolls, there is a slightly better than 65% chance that the game will be over.

Note that this game is incredibly simple, but much more complicated situations can be modeled this way. For example, imagine a game where some of the squares are marked, “Go to jail”, or “Go forward 4 steps”. All that would be affected would be the numbers in the array  $M$ . For a board with 100 positions, the array would be  $100 \times 100$ , but the idea is basically the same.

Suppose you are designing a board game for very young children. You would like to make sure that the game is over in fewer than, say, 50 moves, so you could simply make an array corresponding to a proposed board, iterate as above, and make sure that the game is very likely to be over in 50 moves.

## 15 Final Beautiful Examples

### 15.1 Newton’s Method for Complex Roots

In Section 10 we learned how Newton’s method can be used to find the roots of functions as long as we are able to calculate the slopes of the functions at any point on the curve. In this final section, we are going to do the same thing, but instead of restricting ourselves to the real numbers, we are going to seek roots in the complex plane.

We will use Newton’s method to find the roots of  $x^3 = 1$  – in other words, we will find the cube root of 1. On the surface this seems silly, because isn’t the cube root of 1 equal to 1? Well, it is, but it turns out that 1 has three different cube roots. Here’s why:

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

If we set that equal to zero, we get a root if either  $x - 1 = 0$  or if  $x^2 + x + 1 = 0$ . The first equation yields the obvious root  $x = 1$ , but the second can be solved using the quadratic formula to obtain two additional roots:

$$x = \frac{-1 + \sqrt{3}i}{2} \quad \text{and} \quad x = \frac{-1 - \sqrt{3}i}{2},$$

where  $i$  is the imaginary  $\sqrt{-1}$ . If you haven’t seen this before, it’s a good exercise to cube both of those results above to verify that the result in both cases is 1.

If you plot the three roots, they lie on an equilateral triangle centered at the origin.

Since we will be working with complex numbers, we will change the variable names to be in terms of  $z$  instead of  $x$ . This is not required, but it is the usual convention, and it constantly reminds us that the variables are not necessarily restricted to be real numbers.

As we derived in Section 10, if  $z_0$  is an initial guess for the cube root, the next approximation,  $z_1$ , can be obtained as follows:

$$z_1 = \frac{2z_0^3 + 1}{3z_0^2}.$$

(Actually, this is not exactly the same, since we are trying to find the cube root of 1 instead of 2 as we did in the previous section, but the only difference is that the constant 2 in that section is replaced by a 1 here.)

The other difference, of course, is that we want to allow  $z_0$  and  $z_1$  to take on complex values instead of just real ones. The calculations are a bit messy, but straightforward (see Appendix A for details). As an example, consider performing the iteration above when the initial value is  $.9 + .4i$ :

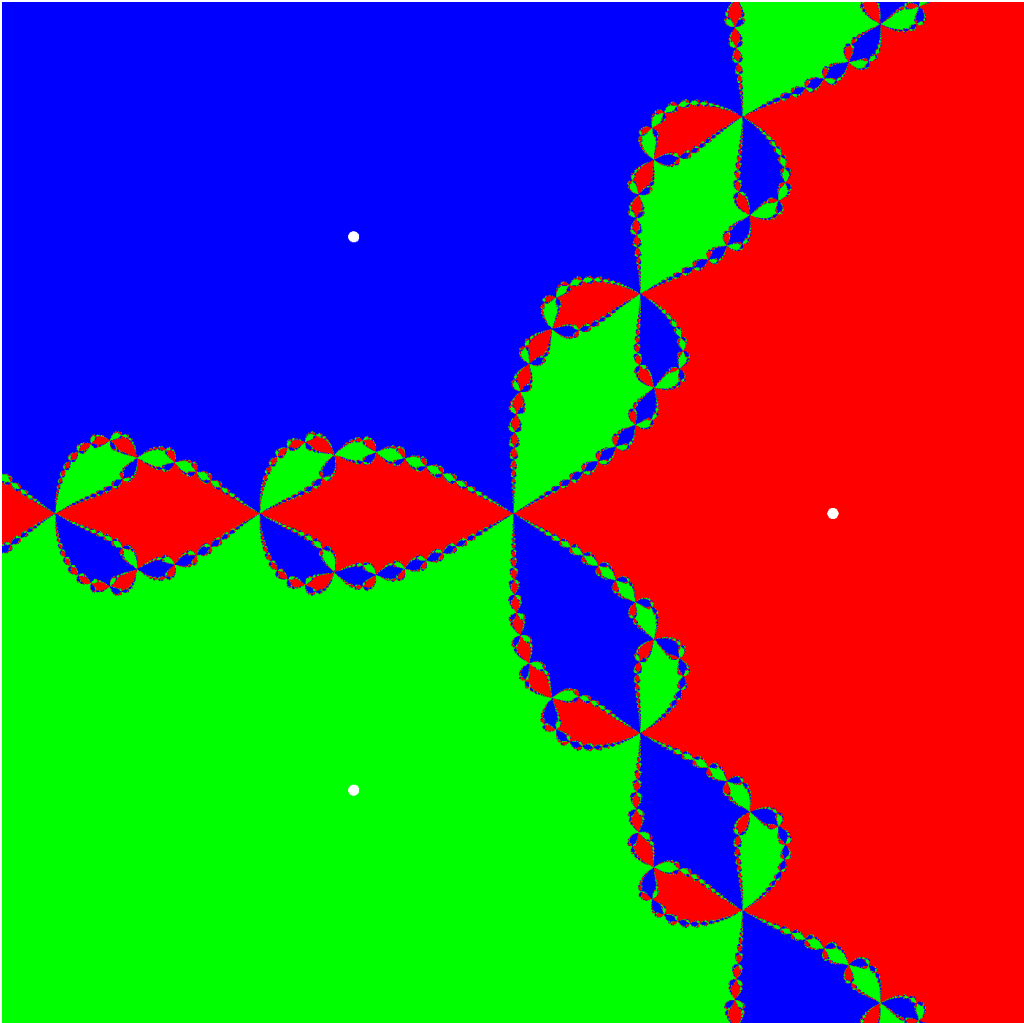
$$\begin{aligned} z_0 &= 0.9 + 0.4i \\ z_1 &= 0.830276 + 0.0115917i \\ z_2 &= 1.03678 - 0.00576866i \\ z_3 &= 1.00126 - 0.000395117i \\ z_4 &= 1.00000, -.0000009935i \end{aligned}$$

It is fairly clear that the sequence above converges to the root  $z = 1$ . Let's try one more, beginning at  $-1 + i$ :

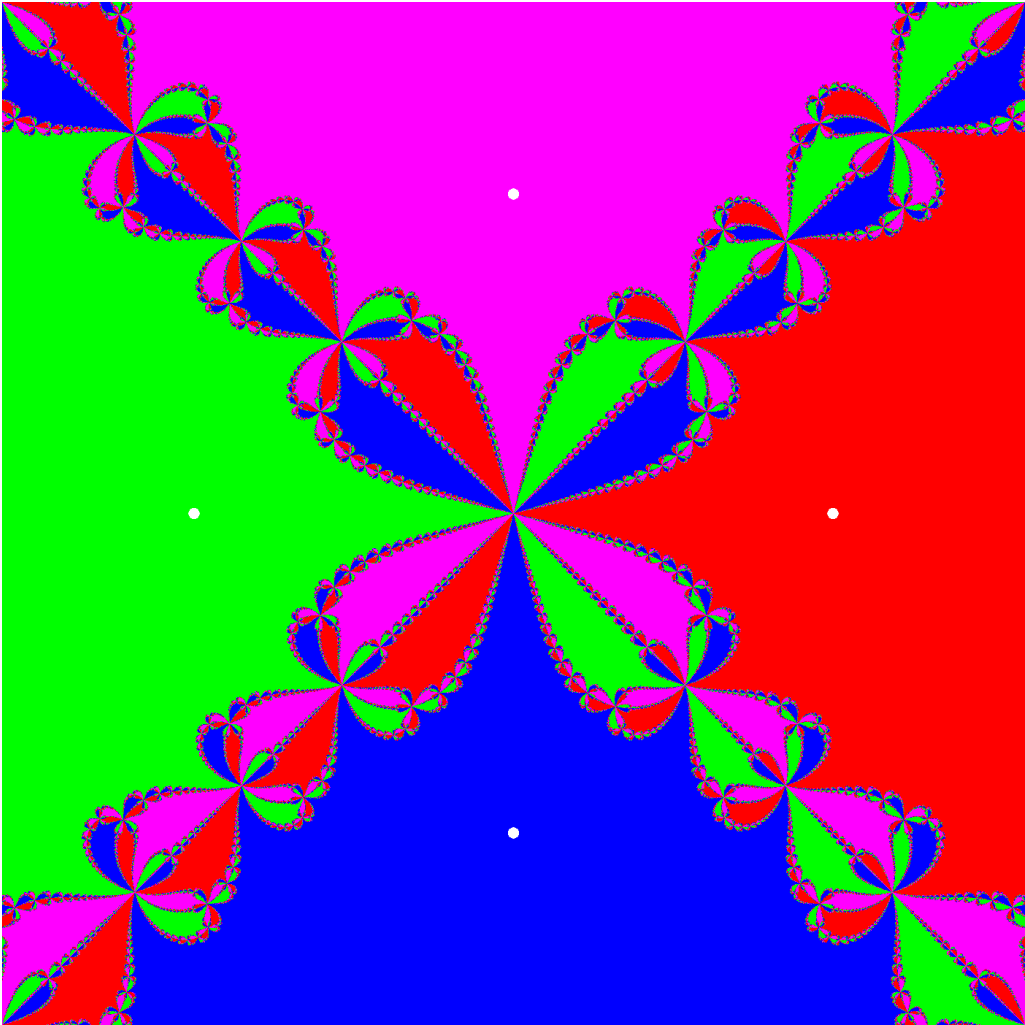
$$\begin{aligned} z_0 &= -1.0 + 1.0i \\ z_1 &= -0.666667 + 0.833333i \\ z_2 &= -0.508692 + 0.8411i \\ z_3 &= -0.49933 + 0.866269i \\ z_4 &= -0.5 + 0.866025i \end{aligned}$$

It is clear that this one converges to  $z = (-1 + \sqrt{3}i)/2 = -.5 + .866025i$ .

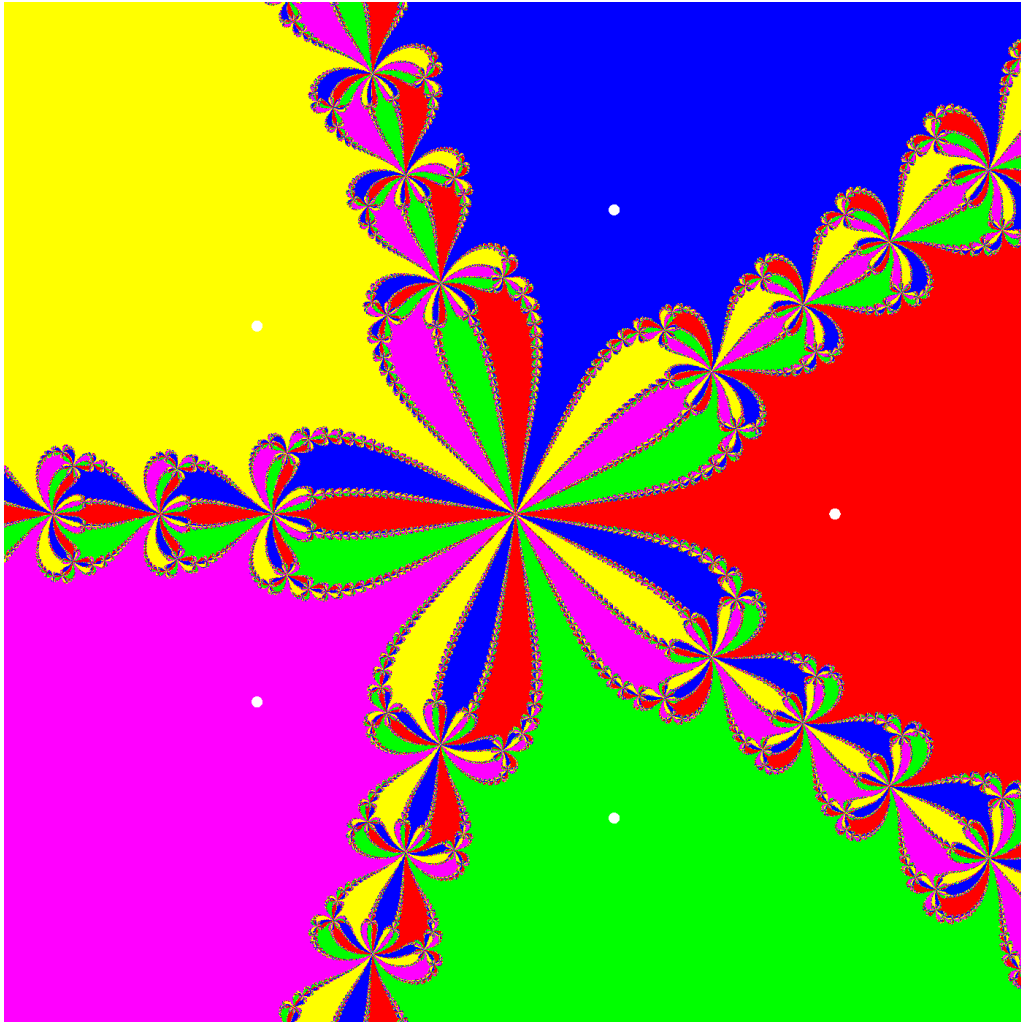
What makes Newton's method in the complex plane interesting is that many initial values lead to multiple interesting jumps around the plane before they converge to one of the roots. In fact, if we look at every point in the plane and color it red if it converges to 1, color it green if it converges to  $(-1 - \sqrt{3})/2$  and blue if it converges to  $(-1 + \sqrt{3})/2$ , then the region around the origin ( $-1.6 \leq x, y \leq 1.6$ ) would be colored as in the following illustration. These are the domains of attraction in the complex plane for Newton's method applied to this polynomial. The white dots in the red, green and blue regions represent the three roots of the equation  $z^3 - 1 = 0$ .



Just because it's another pretty picture, here is another image which corresponds in exactly the same way as the previous one to the use of Newton's method to solve the equation  $z^4 - 1 = 0$ . This time there are four roots:  $1$ ,  $-1$ ,  $i$  and  $-i$ , and the colors correspond to regions of input values that will eventually converge to those roots.



Finally, here's exactly the same thing, but solutions of  $z^5 - 1 = 0$ :



## 15.2 The Mandelbrot Set

Newton's method iteratively applies a function to attempt to converge on a root, but other sorts of iteration can lead to different artistic results. One of the most famous is called the Mandelbrot set.

The definition of the Mandelbrot set is simple. Consider the function:  $f(z) = z^2 + c$ , where  $z$  and  $c$  are complex numbers. For any particular fixed  $c$ , we consider the iterates:  $f^{(0)}(0)$ ,  $f^{(1)}(0)$ ,  $f^{(2)}(0)$ ,  $f^{(3)}(0)$ ,  $\dots$ . In other words, iterated the function  $f$  beginning with the value 0. If the absolute value of  $z$  ever gets larger than 2, successive iterates of  $f$  from then on increase in size without bound. Another way to say that is that the iterates "diverge to infinity".

Thus the series of iterates will always diverge if  $|c| > 2$ , but for some values of  $c$  this divergence does not occur. An obvious example is  $c = 0$ , in which case  $f^{(n)}(0) = 0$ , but there are a lot of others. For example, if we let  $c = 0.3 + 0.3i$ , then the

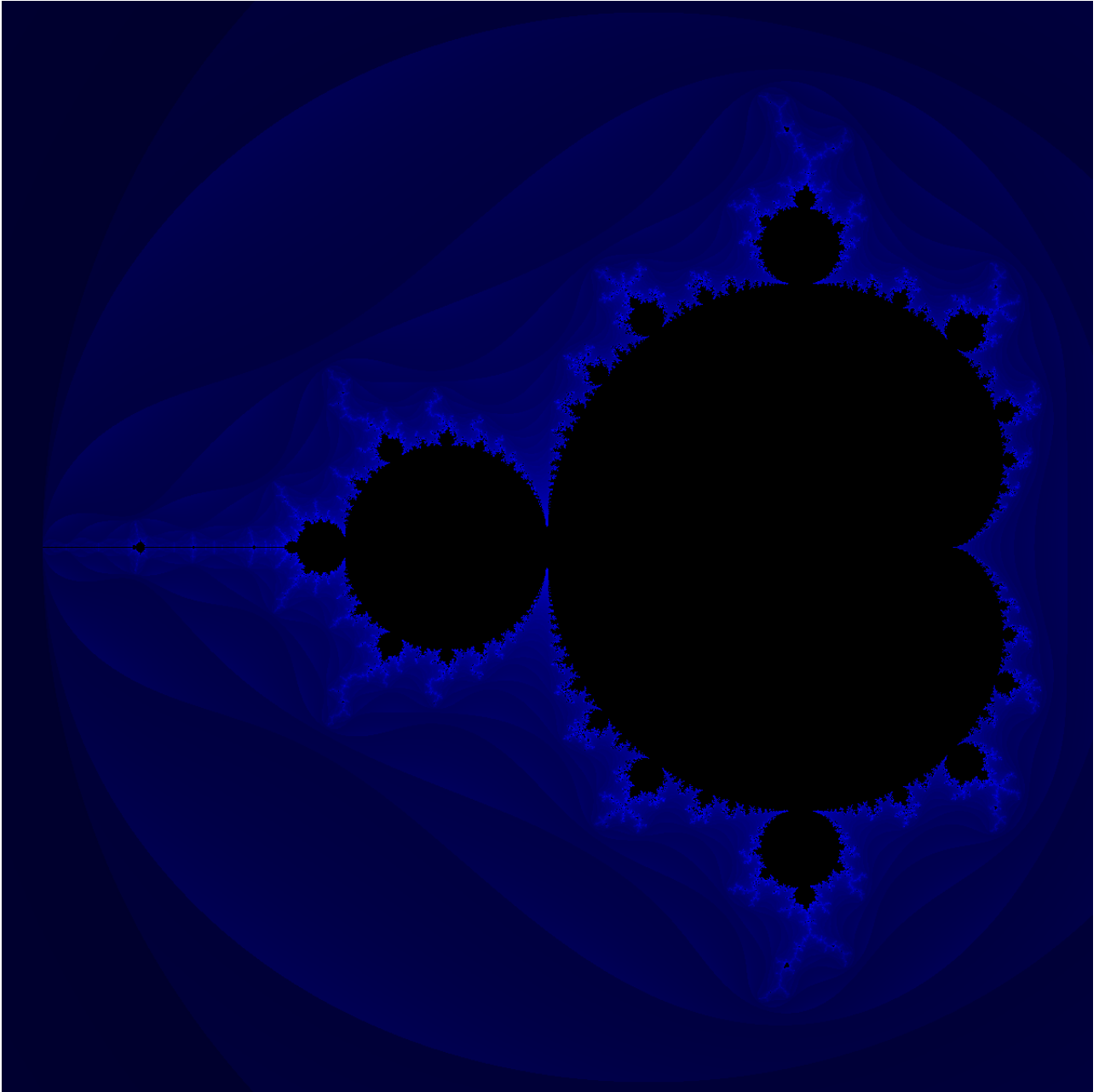


first few iterations, beginning with zero, yield:

$$\begin{aligned}f^{(0)}(0) &= 0.00000 + 0.00000i \\f^{(1)}(0) &= 0.30000 + 0.30000i \\f^{(2)}(0) &= 0.30000 + 0.48000i \\f^{(3)}(0) &= 0.15960 + 0.58800i \\f^{(4)}(0) &= -0.02027 + 0.48769i \\f^{(5)}(0) &= 0.06257 + 0.28023i\end{aligned}$$

The sequence above wanders around for a long time, but eventually converges to  $0.143533 + 0.420796i$ , which you can check to be a solution for  $f(z) = z$ , when  $c = 0.3 + 0.3i$ .

The Mandelbrot set is simply the set of complex numbers  $c$  for which the series of iterates of  $f$ , beginning at zero, do not diverge. The following is an illustration of the Mandelbrot set in the complex plane, where the members of the set itself are drawn in black. The somewhat artistic blue outline of the Mandelbrot set is an indication of how quickly those points diverge. The ones that diverge the slowest (and are, in some sense, closest to the boundary of the Mandelbrot set) are painted in the brightest color.



## A Iterates for Newton's Method

To iterate the function representing Newton's method for the solution in the complex plane of the equation  $z^3 - 1 = 0$ , we must be able to compute:

$$z_1 = \frac{2z_0^3 + 1}{3z_0^2},$$

where  $z_0$  is given.

We write  $z_0 = x + yi$ , where  $x$  is the real part of  $z_0$  and  $y$  is the imaginary part. Then we have:

$$\begin{aligned} z_1 &= \frac{2(x + yi)^3 + 1}{3(x + yi)^2} \\ z_1 &= \frac{2(x^3 + 3x^2yi - 3xy^2 - 3y^3i) + 1}{3(x^2 + 2xyi - y^2)} \\ z_1 &= \frac{(2x^3 - 6xy^2 + 1) + (6x^2y - 2y^3)i}{(3x^2 - 3y^2) + (6xy)i} \end{aligned}$$

The fraction above has the form:

$$z_1 = \frac{a + bi}{c + di},$$

where  $a = 2x^3 - 6xy^2 + 1$ ,  $b = 6x^2y - 2y^3$ ,  $c = 3x^2 - 3y^2$  and  $d = 6xy$ . We have:

$$\begin{aligned} z_1 = \frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ z_1 &= \frac{(ac + bd) + (bc - ad)i}{(c^2 + d^2)} \\ z_1 &= \left( \frac{ac + bd}{c^2 + d^2} \right) + \left( \frac{bc - ad}{c^2 + d^2} \right) i \end{aligned}$$

and the final line shows us how to calculate the real and imaginary parts of  $z_1$  in terms of the real and imaginary parts of  $z_0$ .

Let's illustrate this with  $z_0 = -1 + i$  which was an example in Section 15. We have  $x = -1$  and  $y = 1$ . We then have:

$$z_1 = \frac{(-2 + 6 + 1) + (6 - 2)i}{(3 - 3) - 6i}.$$

So  $a = 5$ ,  $b = 4$ ,  $c = 0$  and  $d = -6$ , yielding:

$$z_1 = \frac{-24}{36} + \frac{30}{36}i = -\frac{2}{3} + \frac{5}{6}i,$$

which is what we obtained earlier.