Inversion in a Circle

Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
March 23, 2011

Abstract

This article will describe the geometric tool of inversion in a circle, and will demonstrate how it can be used. Proofs of the properties of inversion will usually not be included.

1 Introduction

In this document, some of the mathematics is not absolutely rigorous, but can be made so. The main purpose of the document is to provide an intuition about inversion and to illustrate how the method can be used in many situations to convert difficult problems into simpler ones.

Inversion in a circle is a method to convert geometric figures into other geometric figures. It is similar to reflection across a line:

- Any figure can be reflected across a line or inverted in a circle.
- Reflecting a figure across the same line twice returns it to its original form. The same is true for inversion in a circle.
- Reflection takes points to the other side of the line; inversion takes points to the "other side" of the circle. In other words points inside are inverted to the outside and vice-versa.
- There is a fairly easy mathematical relationship between a figure and its reflection or between a figure and its inversion.
- Sometimes it is much easier to work with the reflected version or the inverted version of a figure.

There is a simple way to describe how a point can be inverted in a circle. If we wish to invert a more complex figure than a single point, we simply invert every point in the figure and the resulting set of points becomes the inverted figure.

In this document, we will describe some ways to think about inversion that may not be mathematically perfect, but they provide some good intuition about inversion that will usually lead you to correct conclusions.

It turns out that (loosely speaking), “circles” invert to “circles”, but for now we need to include the quotation marks, since by “circles” we need to include “special” circles that have an infinite radius. In other words, we will consider straight lines to be a special type of “circle”.

2 Basic Definition of Inversion

In the same way that reflection across a line depends on the particular line you choose, inversion in a circle depends on the particular circle. The basic definition of inversion of a point in a circle is simple:

If k is a circle with center O and radius r, and P is any point other than O, then the point P' is the inversion of P if:

- P' lies on the ray $\overrightarrow{OP}$.
- $|OP| \cdot |OP'| = r^2$.

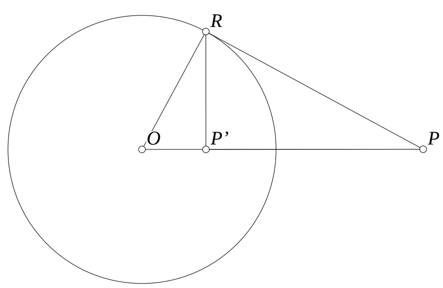

Figure 1: Inversion of a point

Figure 1 shows the inversion of a typical point and how such a point could be constructed with straightedge and compass. If P is outside the circle k, construct a tangent line from P to k touching k at R. Drop the perpendicular from R to the ray $\overrightarrow{OP}$ and P', the inversion of P, is at the point of intersection of that perpendicular and the ray. If P' is inside the circle do the opposite: construct the perpendicular to $\overrightarrow{OP'}$ at P' which intersects the circle k at R. The tangent to k at R intersects the ray $\overrightarrow{OP'}$ at P.

This figure and the description in the previous paragraph shows that inverting a point twice with respect to the same circle returns the point to its original position.

To show the relationship of the lengths, since $\triangle OP'R \sim \triangle ORP$, we have:

$$|OP'|/|OR| = |OR|/|OP|,$$

and since $|OR| = r$, this is equivalent to:

$$|OP| \cdot |OP'| = r^2.$$

Notice that if P happens to lie on the circle k, then the inversion of P is the same as P.

3 The "Point at Infinity"

Notice that the using the definition in Section 2 we can invert any point on the plane other than the point O itself.

It is easy to see why there is a problem when we try to invert the point O. If we consider points P that are very close to O, the length $|OP|$ will be tiny, and to satisfy the equation $|OP| \cdot |OP'| = r^2$ will require that the length $|OP'|$ will have to be huge. You can also visualize what happens with the geometric construction displayed in Figure 1: as the point P' moves toward O the point R moves such that the line RP is closer and closer to parallel with the ray $\overrightarrow{OP'}$. When they are parallel, the intersection does not exist. It's as if the point P' moves infinitely far away along the line.

It's a pain that there is a point that cannot be inverted, and one way out of the problem mathematically is to decide that we are no longer considering a standard Euclidean plane, but rather an extended plane that includes a single "point at infinity" which is the inversion of the center of the circle. In order that repeating the inversion twice brings every point back to its original position, we simply say that the inversion of the point at infinity though any circle is the center of that circle.

The plane extended in this way has only one point at infinity, and it's nice to think of it as a place you approach if you move farther and farther away. For that reason, it is useful to think of that point as being on every straight line in the plane, since every line eventually gets arbitrarily far from any particular point on the plane.

If we think about these new "lines" that are the same as the old lines but with an additional point at infinity, then we can approach that additional point by traveling along that original line in either direction. If we could get to that point and go past it, it would appear that the line looped around "through infinity" and came back from infinity from the other direction. Or in other words, these new lines behave like loops, or circles.

So from this point on in this document, when we use the word "circle" with quotation marks around it, we will always mean either a normal circle in the plane or one of these lines that has been augmented with the point at infinity so that it behaves somewhat like a circle. If there are no quotation marks, we will just mean a normal Euclidean circle. We will always put the quotation marks around "lines" since we will always want to include that point at infinity.

4 Inversion of "circles"

With this enhanced idea of a "circle", a key property of inversion is the following: If every point on a "circle" is inverted through a circle k, the result will be a "circle".

The statement above is not hard to prove, but it takes time and there are lots of cases to consider, so we will take it on faith, but we will list here some key results of inversion with respect to a circle k with center O, each of which requires a proof that's not included here:

- Circles completely inside of k that do not pass through O are inverted to circles completely outside of k and vice-versa. Circles that intersect k not passing through O will invert to circles that also intersect k at the same point(s).
- Circles that pass through O are inverted to "lines". If that circle also passes through k at two points P and Q, its inversion will be the "line" passing through P and Q. If a circle passes through O and is internally tangent to k, its inverse will be the "line" externally tangent to k.
- A "line" that passes through O is inverted to itself. Note, of course that the individual points of the "line" are inverted to other points on the "line" except for the two points where it passes through k.
- Every "line" that does not pass through O is inverted to a circle (no quotes: a real circle) that passes through O.
- Two "Circles" that intersect in zero, one, or two places are inverted to other "circles" that intersect in the same number of places. A little care must be taken to interpret this statement correctly if the intersection or tangency is at O. For example, if two circles are tangent at O, then their inverses will be two parallel "lines" (that "meet at infinity"). If a line and a circle are tangent at O, then the inverse of the circle will be parallel to the line (which is inverted into itself).
- A "circle" intersecting or tangent to k is inverted to a "circle" intersecting or tangent to k in exactly the same places or place.
- We can define the angle between two "circles" by finding the angle between the lines tangent to the "circles" at the point of intersection. Of course if the "circle" is a line, just consider the tangent line to be the line itself. Tangent circles make an angle of 0 as to parallel "lines" that "meet at infinity". Using this definiton, one final property of inversion is this: if two "circles" meet at an angle α, then their inversions also meet at the same angle α.

For the rest of this document, we will assume without proof that all the statements above are true, and we will demonstrate how the tool of inversion can be applied to solve a variety of problems.

5 Geometric Constructions

First, note that using the construction illustrated in Figure 1 we can invert any point other than the center of k. So if we wish to invert a circle that does not pass through O, we can just invert any three points on that circle and construct the circle passing through the three inverted points. If the circle does pass through O, invert any two points on the circle and draw the line passing through the inverted points, et cetera.

For our first application, consider the problem of constructing the circle or circles that pass through two given points P and Q and is tangent to a given line l that does not pass through both P and Q. If l passes through one of them, let that one be P, and let Q be the other.

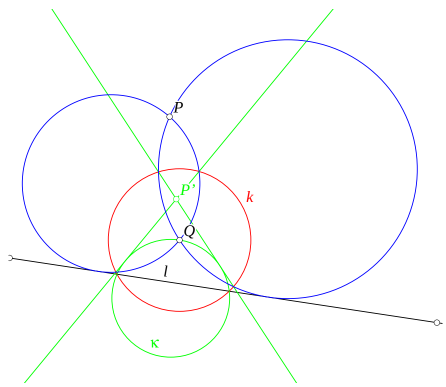

Figure 2: Circle tangent to l through P and Q

(See Figure 2) Let k be any circle centered at Q (colored red in the figure) and invert P and l through k. The circle (or circles) we are seeking passes through P which is the center of k, so its inversion will be a "line". The point P will invert to a new point P' (which will not be at infinity, since P and Q are different). The line l will invert to some circle which we'll call κ.

The "line" which is the inversion of the circle(s) we are seeking is tangent to κ and passes through P'. If P' lies on κ (meaning P lies on l), then there is one solution: the tangent to κ at P'. Otherwise, there are either two lines through P' that are tangent to κ (the external tangents), or none (if P' lies inside κ). If we invert this line or these lines (drawn in green) through k we obtain the solution(s) which are colored blue in the figure.

Since this is our first example, let's look carefully at what occurred in this particular arrangement. Note the following:

- In this example, the line l passes through the circle k so its inversion, κ, also passes through k at the same two points.
- The inversion through a circle of any line is a circle that passes through the center of the circle of inversion. In this example, Q is the center of the circle of inversion, and the circle κ passes through Q.
- Since P is outside the circle k its inversion, P', lies inside of k.
- In this example, the two tangent lines to κ through P' pass through k so their inversions (the blue circles) will pass through k at the exact same points.

Now let's change a few things in the figure and see how that affects the solution. For the first example, see Figure 3, where the only thing that is changed is the size of the circle of inversion. This should result in exactly the same solutions (and you can see that it does by comparing it to Figure 2)

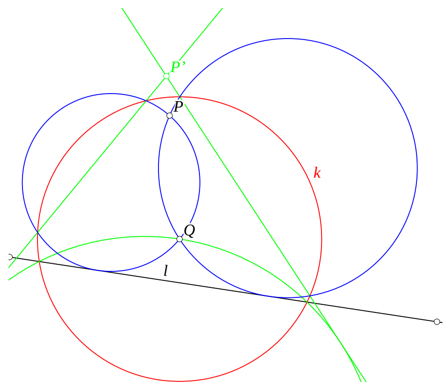

Figure 3: Circle tangent to l through P and Q

In the next example (Figure 4), note the differences, but also notice things that are the same:

- The line l is outside k, so κ is completely inside k.
- The points P and Q have moved relative to l so that one of the solution circles is now very large. If the line through P and Q were parallel to l, one of the solution "circles" would be a line parallel to l.

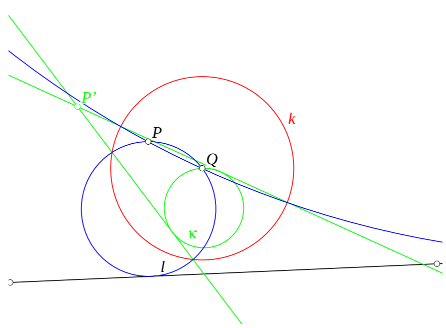

Figure 4: Circle tangent to l through P and Q

When are there no solutions? For this to occur, P' would have to lie inside the circle κ and this means that P and Q were on opposite sides of l and in that case there are obviously no solutions.

As a second example, consider a similar problem: construct the circle (or circles) tangent to two given circles k_1 and k_2 and passing through a point P. If only P were at infinity, those lines would just be the common tangents to the circles, right? So all we need to do is invert everything relative to a circle k that is centered at P. Then we find the common tangent lines to the resulting inverted circles and re-invert them in k to obtain our solution(s).

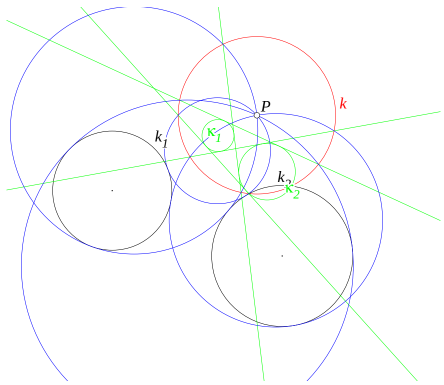

Figure 5: Circle through P tangent to k_1 and k_2

In Figure 5 we see the complete construction. The circle of inversion (k) is in red, the inversions of circles k_1 and k_2 (called κ_1 and κ_2) are in green. The four common tangents to κ_1 and $\kappa_@$ (both internal and external) are in green, and the four blue circles are obtained by re-inverting those four lines in k. You can see that all four circles pass through P and are tangent to k_1 and k_2.

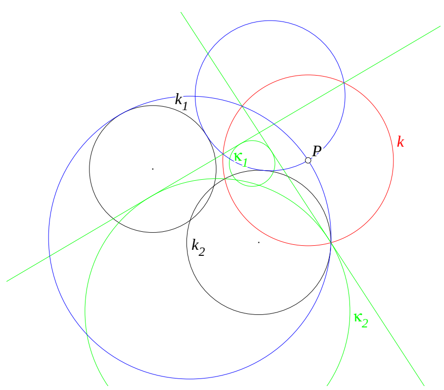

Figure 6: Circle through P tangent to k_1 and k_2

If k_1 and k_2 intersect, so will κ_1 and κ_2 so there will be no common internal tangents and therefore there will be only two solutions. See Figure 6.

In Figure 7 the circle k_2 is inside k_1 and there are still four solutions. If P is inside

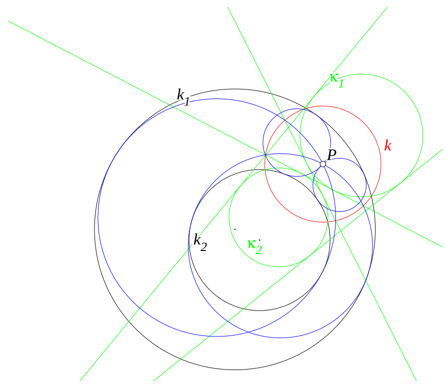

Figure 7: Circle through P tangent to k_1 and k_2

both or outside both, there will be no solutions.

6 Ptolemy's Theorem

Ptolemy's Theorem says that in any cyclic quadrilateral $ABCD$ that:

$$|AC| \cdot |BD| = |AD| \cdot |BC| + |AB| \cdot |CD|.$$

The quadrilateral $ABCD$ is said to be cyclic when A, B, C and D all lie on the same circle.

Here is a proof of Ptolemy's Theorem using inversion in a circle. See Figure 8.

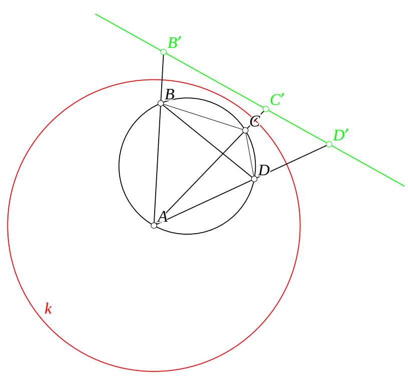

Figure 8: Ptolemy's Theorem

Let k be a circle centered at A and invert all four points on the quadrilateral $ABCD$ and the circle upon which they lie with respect to k. Since the circle passes through A which is the center of the circle of inversion, the circle will be inverted to a straight line (green in the figure). The point A will be inverted to the point at infinity, but B, C and D will be inverted to points B', C' and D', all lying on the (green) line.

By the definition of inversion, we have:

$$|AD| \cdot |AD'| = |AC| \cdot |AC'| = |AB| \cdot |AB'| = r^2,$$

where r is the radius of the circle k.

We will show that $\triangle ABC \sim \triangle AC'B'$. From the equation in the last paragraph, we have $|AB|/|AC| = |AC'|/|AB'|$ and since $\angle A$ is equal to itself, by SAS similarity, we have $\triangle ABC \sim \triangle AC'B'$. Exactly the same argument can be used to show that $\triangle ADC \sim \triangle AC'D'$.

By similarity, we have:

$$\frac{|B'C'|}{|BC|} = \frac{|AB'|}{|AC|}.$$

Thus:

$$|B'C'| = \frac{|BC| \cdot |AB'|}{|AC|},$$

and since $|AB| \cdot |AB'| = r^2$ we have:

$$|B'C'| = \frac{|BC| \cdot r^2}{|AC| \cdot |AB|}. \tag{1}$$

Exactly the same argument shows us that:

$$|B'D'| = \frac{|BD| \cdot r^2}{|AD| \cdot |AB|} \tag{2}$$

and

$$|C'D'| = \frac{|CD| \cdot r^2}{|AC| \cdot |AD|}. \tag{3}$$

But B', C' and D' lie on a line, so we know that

$$|B'D'| = |B'C'| + |C'D'|. \tag{4}$$

Substituting Equations 1, 2 and 3 into 4, we obtain:

$$\frac{|BD| \cdot r^2}{|AD| \cdot |AB|} = \frac{|BC| \cdot r^2}{|AC| \cdot |AB|} + \frac{|CD| \cdot r^2}{|AC| \cdot |AD|},$$

and if we multiply through by $(|AB| \cdot |AC| \cdot |AD|)/r^2$ we obtain the final result:

$$|AC| \cdot |BD| = |AD| \cdot |BC| + |AB| \cdot |CD|.$$

By the way, if the quadrilateral inscribed in a circle happens to be a rectangle, then we can use Ptolemy's theorem (proved above completely using inversion) to prove the Pythagorean theorem, so in a sense, the Pythagorean theorem can be proved using inversion in a circle.

7 Miquel’s Theorem

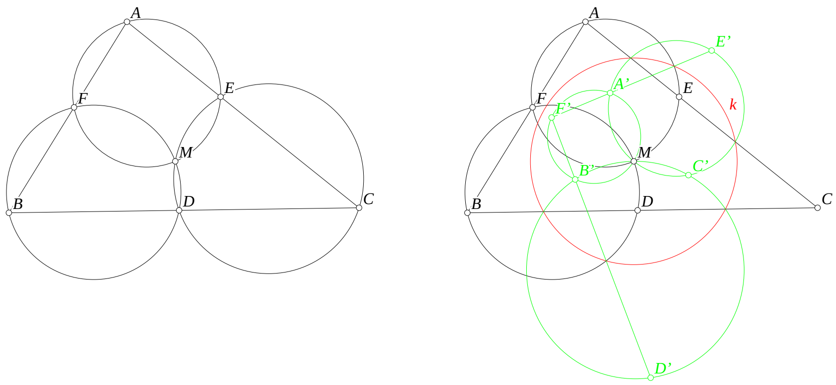

Figure 9: Miquel’s theorem

See Figure 9. On the left side of the figure, let $\triangle ABC$ be an arbitrary triangle and let D,E and F be arbitrary points on the lines BC, CA and AB, respectively. (In the diagram those points lie on the segments, but they can also lie outside the segments and the theorem continues to hold.) Miquel’s theorem states that the circles passing through AFE, through BFD and through CDE are all concurrent at some point M.

To prove the theorem, consider the diagram on the right of Figure 9. Let the circles passing through AFE and BFD meet at a point M in addition to at point F. We will show that M lies on the circle CDE.

Choose an arbitrary circle k centered at M and invert all the points in k so that A inverts to A', et cetera. By the properties of inversion, $F'A'E'$ lie on a line, as do $F'B'D'$. Since the point M is inverted to the point at infinity, the point at infinity is inverted back to M, and since the point at infinity is on all the lines that make up the sides of the triangles, we know that $A'F'B'M$, $B'D'C'M$ and $A'E'C'M$ are each a cyclic set of points. If we can show that D', C' and E' are collinear, then by re-inverting that line through k we will arrive at the circle $CEDM$ which will prove our theorem.

Let $\angle A'F'B' = \alpha$, $\angle A'E'C' = \beta$ and $\angle B'D'C' = \gamma$. Since opposite angles in a cyclic quadrilateral add to π, the angle around point M will be $(\pi-\alpha)+(\pi-\beta)+(\pi-\gamma) = 2\pi$, so $\alpha+\beta+\gamma = \pi$. But if C' lies on one side or the other of $D'E'$ then $\alpha+\beta+\gamma$ will not equal π, so C' lies on $D'E'$ and we are done.

8 Peaucellier’s Linkage

For many years, it was unknown whether it was possible to construct a mechanical linkage that would turn perfect circular motion into perfect linear motion. Peaucellier’s

Linkage (see figure 10) shows a linkage that achieves this conversion.

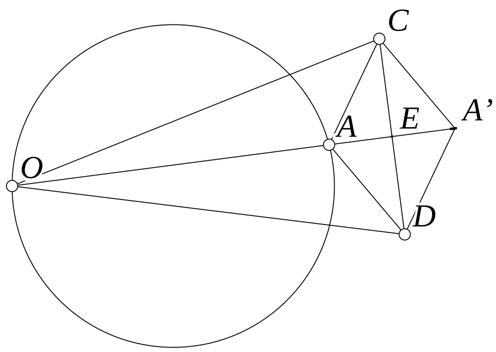

Figure 10: Peaucellier's Linkage

In the linkage, point O is fixed on the circle, point A is constrained to move on the circle, and segments OC and OD are two bars of the length l, while segments AC, CA', $A'D$, and DA are four bars of length r. The bars are all hooked together with flexible joints at points O, A, C, A', and D. (The lines OA' and CD in the figure are solely for the proof—they are not part of the linkage.)

We can show that the point A' will lie on a straight line if we can show that $OA \cdot OA'$ is constant. If that is the case, then A and A' are inverse points with respect to a circle centered at O. As the point A moves on a circle that passes through O, its inverse, A' must move along the inverse of that circle, which is a straight line since O lies on the circle upon which A is constrained to lie. (If O is not on the circle, A and A' will still be inverse points relative to a circle centered at O, but A' will merely move on a different circle as A traces out the first one.

To show this, construct the lines OA' and CD. Since $ACA'D$ is a rhombus, $CE \perp OA'$ and E bisects AA'. Thus we have

$$OA \cdot OA' = (OE - AE)\cdot(OE + EA') = (OE - AE)\cdot(OE + AE) = OE^2 - AE^2.$$

Using the pythagorean theorem on $\triangle AEC$ and on $\triangle OEC$, we have

$$\begin{aligned} OE^2 + EC^2 &= OC^2 = l^2 \\ AE^2 + EC^2 &= AC^2 = r^2. \end{aligned}$$

Subtracting, we obtain:

$$OE^2 - AE^2 = l^2 - r^2 = OA \cdot OA'.$$

Since $l^2 - r^2$ is constant, so is $OA \cdot OA'$ and we are done.

9 The Steiner Porism

We will use inversion in a circle to prove an amazing property of pairs of circles, one inside the other.

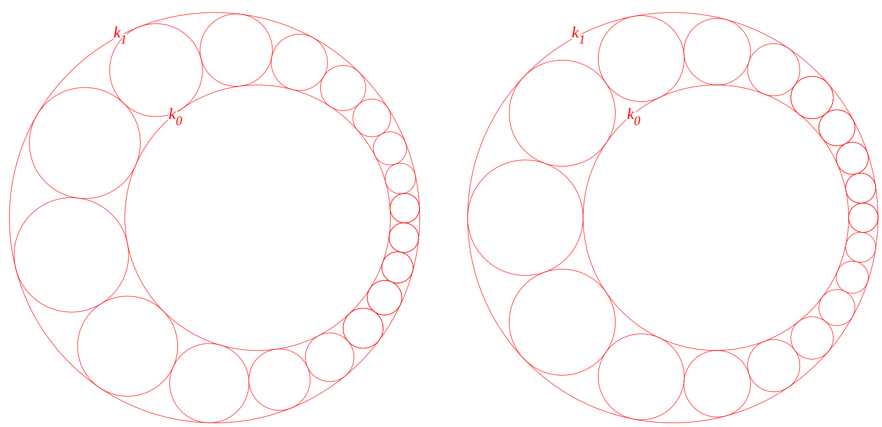

Figure 11: The Steiner Porism

See Figure 11. Consider two circles k_0 and k_1, where k_0 lies inside k_1, but not necessarily centered inside. If you draw any circle between the two as in the figure, and then continue to draw a series of circles that are tangent to k_0 and k_1 and also to the previous circle you drew, one of two things happen. Either the final circle you draw is also tangent to the original circle (as in the figure) or it is not. The amazing thing is that if you achieve tangency with some choice of a starting circle, you will achieve tangency with any such choice. Equivalently, if you fail to achieve tangency with your first choice, you will never achieve tangency with any other choice.

Figure 11 shows an example of k_0 and k_1 where tangency occurs all the way around and illustrates two different rings of circles with different starting points.

On the other hand, the result is obvious if the two circles are concentric as in Figure 12. Since the distance between the circles is constant, every starting position is equivalent to every other starting position since you can just rotate the figure to make them coincide.

We can prove the result by noting that if we do any inversion of a diagram like Figure 11 the result will look somewhat the same: two circles with a ring of other circles between them. Thus, if we can find, for any two circles like k_0 and k_1, an inversion that makes the images of k_0 and k_1 concentric, we will be done, since either the circles between the concentric ones always match up or they never match up.

So all we need to do is find a circle of inversion that takes two arbitrary non-intersecting circles into a pair of concentric ones. This can be done as shown in Figure 13

Given any two different circles, there is a line called the radical axis such that from

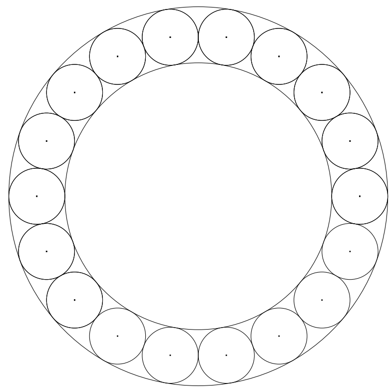

Figure 12: Concentric Circles

every point on the line, tangents to the two circles have the same length. In the figure the green line is the radical axis of circles k_0 and k_1. The black line is the line connecting the centers of those two circles. The radical axis meets the line of centers at R and by the definition of the radical axis, the four external tangents from R to the two circles have the same length, so the green circle centered at R and passing through the tangent points is perpendicular to both circles.

Let k be any circle centered at O, the intersection of that green circle centered at R and the line connecting the centers of k_0 and k_1. If we invert the two circles in k the results (the magenta circles) will have to be perpendicular to two perpendicular lines: the black line connecting the centers of the circles and the inversion of the green circle. Circles simultaneously perpendicular to two perpendicular lines are concentric, so we are done.

Note: If two circles intersect, it is trivial to find the radical axis: it is just the line connecting the two points of intersection. In our case, the two circles don't intersect, so the problem is a bit more difficult. But since every pair of circles has a radical axis, if there are three circles, the radical axes of each pair must meet at the same point called the radical center, and from this one point (which may be at infinity for certain equal-sized circles), the external tangents to all three circles from this point are equal.

So to find the radical axis of two non-intersecting circles, perform the following construction twice: draw a circle that intersects the two in two places, construct their radical axes, and find their radical center. Each of these radical centers lies on the radical axis of the two non-intersecting circles. Connect them with a line and you have the radical axis for the original two circles.

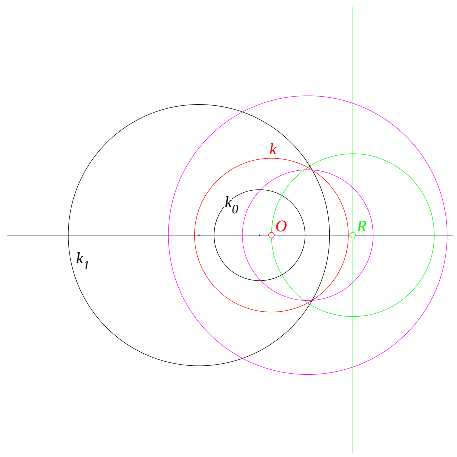

Figure 13: Inversion to Concentric Circles

10 The Arbelos of Pappus

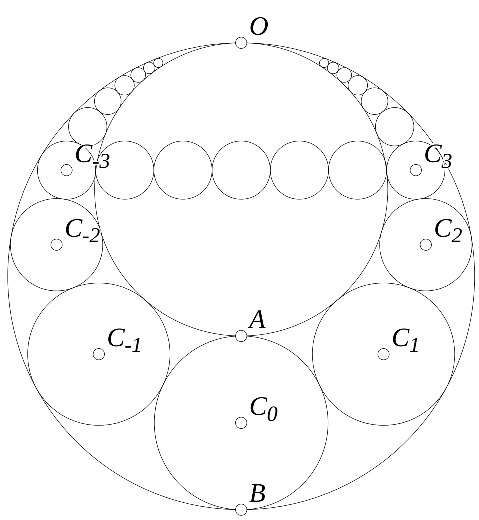

Figure 14: The Arbelos of Pappus

"Arbelos" is the Greek word for a shoemaker's knife. In figure 14, ignore everything except for the three circles with diameters OA, OB, and AB, where O, A, and B lie on the same line, and notice that the area inside the larger circle and outside the two smaller circles is divided into two pieces on the left and right. Either of these shapes, which are basically a half-circle with two half-circles removed, look something like an arbelos.

From those original three circles, construct a series of circles on both sides of the circle with diameter AB as shown in the figure. Each circle is tangent to the circles with diameters OA and OB and also tangent to the previous circle in the series. In the figure, the centers of those circles on the right are labeled C_1, C_2, et cetera, and those on the left have centers C_{-1}, C_{-2}, $\ldots$.

We will show that the distance between C_{-1} and C_1 is twice the diameter of the circle centered at C_1, that the distance between C_{-2} and C_2 is four times the diameter of the circle centered at C_2, and in general, that the distance between C_{-n} and C_n is $2n$ times the diameter of the circle centered at C_n. In the figure, this is illustrated for C_{-3} and C_3 and the circles centered there—exactly 5 circles of the same diameter as those circles can be placed on a straight line between them. There would be 1 between the circles at C_{-1} and C_1, 3 of them between the circles at C_{-2} and C_2, et cetera.

The proof is not difficult, and since we have been looking at inversion and the Steiner porism, it is clear that the situation here is very similar. If, for example, we can find an inversion that leaves the circles centered at C_{-3} and C_3 fixed and at the same time maps the circles with diameters OA and OB into parallel lines, we will be done. The circles between those centered at C_{-3} and C_3, namely those centered at C_{-2}, C_{-1}, C_0, C_1, and C_2, under inversion will remain tangent to their neighboring circles, and to the two parallel lines. Clearly, when we look at C_{-4} and C_4, there will be two more circles in the chain between them, so they will have two more circles on the line between them.

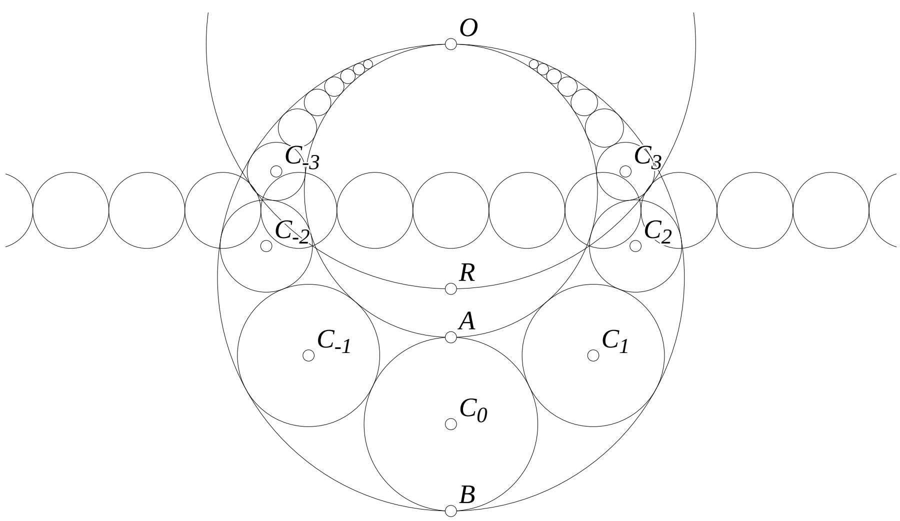

Figure 15: Proof of the Arbelos Property

It is not hard to find such an inversion. Clearly, it will have to send the circles with diameters OA and OB to parallel lines, so the circle of inversion must be centered at O. Figure 15 shows the inversion of the circles C_i in such a circle centered at O and passing through a point R.

Since the circles C_{-i} and C_i are symmetric relative to the line OB, as the radius of the circle of inversion centered at O increases, it will expand through each pair of opposite circles in exactly the same way. At some point as it expands through each pair, it will be orthogonal to both and at that point, both of those circles will be inverted into themselves.

11 A Four-Circle Problem

Four circles to the kissing come,
The smaller are the better.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of squares of all four bends
Is half the square of their sum

Frederick Soddy

See figure 16. Given a line, construct a circle tangent to it centered at C_1 with radius r_1. Next, construct a circle tangent to both the line and the circle centered at C_1. The new circle has center C_2 and radius r_2. As shown in the figure, construct the circle centered at C_3 of radius r_3 tangent to both circles and to the line, as shown. Finally, the circle centered at C_4 of radius $r_4 = 1\ \mathrm{cm}$ is tangent to the first three circles and lies inside as in the figure. Find the perpendicular distance from C_4 to the line in terms of r_1, r_2, r_3, and $r_4 = 1\ \mathrm{cm}$.

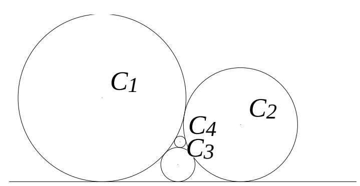

Figure 16: A Four-Circle Problem

Although at first it does not look like it, this is just a special case of the Steiner porism, but in this case, the outer circle is the "circle" with an infinitely large radius—the straight line. Imagine what would happen if this figure were inverted in a random circle that was not special in any way (none of the lines or circles pass though its center). It would become the simplest example of a Steiner porism with an inner and outer circle and three circles filling the ring between them.

If that is the case, it does not matter what r_1, r_2, and r_3 are—the height above the line will only depend on the radius $r_4 = 1\ \mathrm{cm}$.

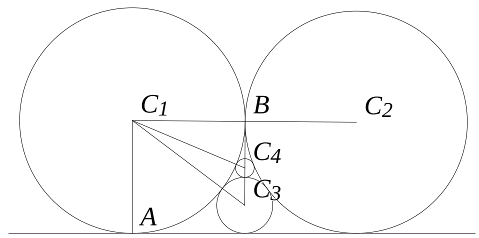

Figure 17: A Four-Circle Problem

So we might as well choose a set of circles with convenient radii as in figure 17, where $r_1 = r_2$.

Using the pythagorean theorem on the right triangle $\triangle C_1BC_3$ we have:

$$\begin{aligned} r_1^2 + (r_1 - r_3)^2 &= (r_1 + r_3)^2 \\ r_1^2 + r_1^2 - 2r_1r_3 + r_3^2 &= r_1^2 + 2r_1r_3 + r_3^2 \\ r_1^2 - 4r_1r_3 &= 0 \\ r_1(r_1 - 4r_3) &= 0 \\ r_1 &= 4r_3. \end{aligned}$$

Now use the pythagorean theorem again, but this time on $\triangle C_1BC_4$:

$$\begin{aligned} r_1^2 + (r_1 - 2r_3 - r_4)^2 &= (r_1 + r_4)^2 \\ r_1^2 + (r_1 - r_1/2 - r_4)^2 &= (r_1 + r_4)^2 \\ r_1^2 + (r_1/2 - r_4)^2 &= (r_1 + r_4)^2 \\ r_1^2 + r_1^2/4 - r_1r_4 + r_4^2 &= r_1^2 + 2r_1r_4 + r_4^2 \\ r_1^2/4 - 3r_1r_4 &= 0 \\ r_1(r_1 - 12r_4) &= 0 \\ r_1 = 12r_4. \end{aligned}$$

But $r_4 = 1 \text{ cm}$, so $r_1 = 12 \text{ cm}$ and $r_3 = 3 \text{ cm}$. The height of C_4 above the line is 7 cm.

The solution above seems pretty good, but there is an even easier way. See figure 18. We can invert to a situation where two of the circles become straight lines, and the calculations become even easier. If the radius of the circle centered at C_4 is 1 and the unknown equal radii of the other two circles are both r, we can see that r satisfies:

$$r^2 + (r - 1)^2 = (r + 1)^2.$$

The solution is $r = 4$ and from the figure it is clear that the point C_4 is 7 cm above the lower line.

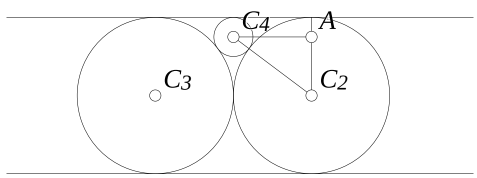

Figure 18: A Four-Circle Problem

Note that this problem could also have been solved using the Descartes circle theorem presented in the form of a poem at the beginning of this chapter. In this case, one of the "circles"—the straight line—has infinite radius, so 1 divided by that radius is zero.

12 Miscellaneous Problems

1. (From BAMO 2008) Point D lies inside the triangle ABC. If A_1, B_1, and C_1 are the second intersection points of the lines AD, BD, and CD with the circles circumscribed about $\triangle BDC$, $\triangle CDA$, and $\triangle ADB$, prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

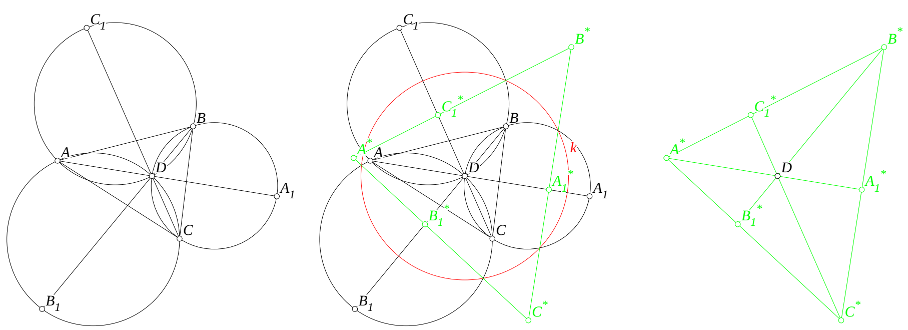

Figure 19: BAMO 2008 Problem

See Figure 19. The original figure is on the left. To solve the problem, draw circle k centered at D having radius 1. Invert the points A, B, C, A_1, B_1 and C_1 with respect to k, yielding points $A^*, B^*, C^*, A_1^*, B_1^*$ and C_1^*, respectively.

Since all three circles pass through D which is the center of inversion, the image of each is a line so the inverse points all lie on the (green) triangle $A^*B^*C^*$.

By the definition of inversion, and the fact that the radius of k is 1, we have: $|AD| \cdot |A^*D| = 1$, $|A_1D| \cdot |A_1^*D| = 1$, and so on.

Consider the first term in the result we are trying to prove: $|AD|/|AA_1|$. We know that $|AA_1| = |AD| + |DA_1|$ so we obtain:

$$\frac{|AD|}{|AA_1|} = \frac{1/|A^*D|}{1/|A^*D| + 1/|A_1^*D|} = \frac{|A_1^*D|}{|A^*D| + |A_1^*D|} = \frac{A_1^*D}{A^*A_1^*}.$$

Doing the same thing for each of the other quotients in the original problem and doing the substitution, the inequality we are seeking to prove becomes:

$$\frac{A_1^*D}{A^*A_1^*} + \frac{B_1^*D}{B^*B_1^*} + \frac{C_1^*D}{C^*C_1^*} = 1.$$

If by $\mathcal{A}(\triangle ABC)$ we indicate the area of triangle ABC, it is clear from the figure that:

$$\frac{A_1^*D}{A^*A_1^*} = \frac{\mathcal{A}(\triangle B^*C^*D)}{\mathcal{A}(\triangle A^*B^*C^*)},$$

and similarly for the other fractions.

Thus our original equality is equivalent to:

$$\frac{\mathcal{A}(\triangle B^*C^*D)}{\mathcal{A}(\triangle A^*B^*C^*)} + \frac{\mathcal{A}(\triangle A^*C^*D)}{\mathcal{A}(\triangle A^*B^*C^*)} + \frac{\mathcal{A}(\triangle A^*B^*D)}{\mathcal{A}(\triangle A^*B^*C^*)} = 1.$$

The three triangles whose areas appear in the numerators together make up the area of the triangle in the denominator, so the problem is solved.

2. Suppose k_1, k_2, k_3 and k_4 are four circles such that k_1 is tangent to k_2, k_2 is tangent to k_3, k_3 is tangent to k_4, and k_4 is tangent to k_1. Show that the four points of tangency lie on a circle or on a straight line. See the diagram on the left in Figure 20.

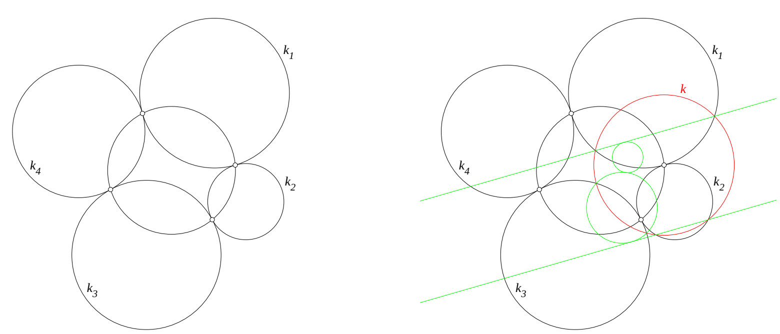

Figure 20: Four Tangent Circles

The proof isn't too hard. Pick any of the tangent points (say the one between k_1 and k_2 and consider a circle k centered at that point. Invert all four circles $k_1, \ldots, k_4$ with respect to k. Since k_1 and k_2 pass through the center of inversion, their inverses will be parallel straight lines. The images of k_3 and k_4 will be two circles tangent to each other and tangent to the parallel lines. See the diagram on the right in Figure 20. It requires only elementary geometry to show that the three tangent points of the green circles and lines lie in a straight line, so their inversion back through k will yield either a circle or a straight line.

3. Let p be the semiperimeter of $\triangle ABC$. Points E and F are on line AB such that $|CE| = |CF| = p$. Prove that the circumcircle of $\triangle CEF$ is tangent to the excircle of $\triangle ABC$ with respect to the side AB. See the diagram on the left of Figure 21.

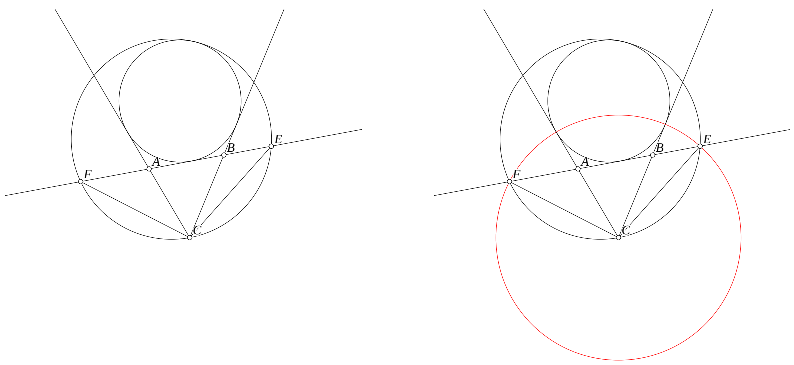

Figure 21: Circumcircle-Excircle Tangency

Invert in a circle centered at C and having radius p. This will leave the points E and F fixed (since they lie on the circle of inversion). It will also leave the excircle fixed, since it is tangent to the lines CA and CB and those are perpendicular to the circle of inversion. On the other hand, the circumcircle of $\triangle CEF$ is the straight line passing through E and F since it passes through the center of inversion. By the definition of the excircle, EF is tangent to it, and thus its inverse relative to the circle of inversion is also tangent to it.

4. (IMO 1985) A circle with center O passes through points A and C and intersects the sides AB and BC of $\triangle ABC$ at points K and N, respectively. The circumcircles of triangles $\triangle ABC$ and $\triangle KBN$ meet at distinct points B and M. Prove that $\angle OMB = 90°$. See the diagram on the left of Figure 22.

Invert through a circle k centered at B. Points A', C' and M' are collinear and so are K', N' and M', whereas $A'C'N'K'$ lie on a circle. We need to find where O' (the image of O) lies. Inversion does *not* map the center of a circle to the center of the inverted circle.

Draw tangents from B to the circle $ACNK$, with tangent points B_1 and B_2. The B_1' and B_2' are the feet of the tangents from B to the circle $A'C'N'K'$ and since

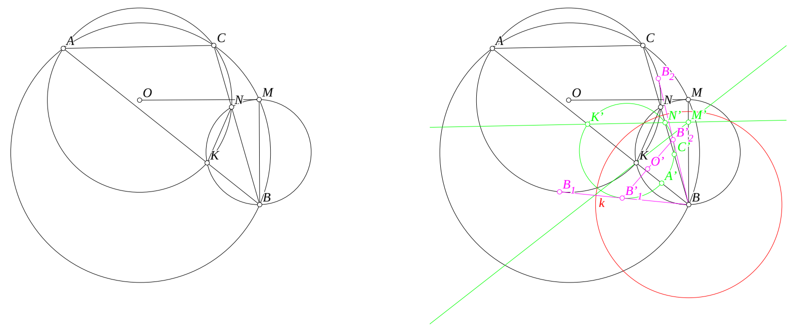

Figure 22: IMO 1985 Problem

L lies on the circle BB_1B_2 its image, O', lies on the line B_1B_2, and in fact is at the midpoint of that line.

Note that M' is on the polar of point B with respect to the circle $A'C'N'K'$, which is the line B_1B_2. Thus $\angle OBM = \angle BO'M' = \angle BO'B'_1 = 90°$.

13 Descartes' Theorem

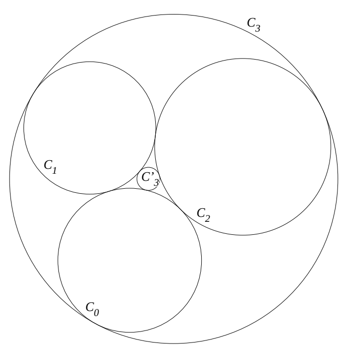

Figure 23: Descartes' Theorem

The following theorem (Descartes' theorem) is a bit difficult to prove, but it will provide some very interesting results related to inversion. See Figure 23. If we have three mutually-tangent circles C_0, C_1 and C_2 having radii r_0, r_1 and r_2, respectively, then there are two other circles tangent to all three (shown in the diagram as C_3 and C'_3). If $k_0 = 1/r_0$, $k_1 = 1/r_1$ and $k_2 = 1/r_2$ then the two solutions for k_3 of the following quadratic equation are the radii of C_3 and C'_3:

$$2(k_0^2 + k_1^2 + k_2^2 + k_3^2) = (k_0 + k_1 + k_2 + k_3)^2. \quad (5)$$

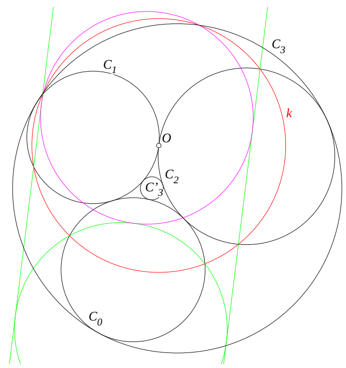

Figure 24: Constructing the tangent circles

Using inversion in a circle, it is fairly easy to construct the two circles C_3 and C'_3. See Figure 24.

Let O be the point of tangency of C_1 and C_2 and let k be the circle of inversion centered at O. If we invert the three circles, we know that C_1 and C_2 must invert to lines since they pass through the center of inversion. We also know that those lines are parallel, since the only place they "meet" is at the image of O, which is inverted "to infinity". Since C_0 is tangent to both C_1 and C_2 its image must be tangent to their images so it must invert to a circle that is tangent to the two parallel lines as in the figure. (In the figure, the images of C_0, C_1 and C_2 are shown in green.)

The circles C_3 and C'_3 are to be tangent to C_0, C_1 and C_2 so their images must be tangent to the images of those three circles. These are easy to construct: they fit between the two parallel green lines and are tangent to both circles. In the figure, only one of those is shown in magenta; the other would be tangent on the other side of the green circle, so it is below the diagram. If we re-invert those two circles in the circle k, we obtain C_3 and C'_3.

Descartes' theorem also holds for circles with "infinite radius", in other words, for straight lines. This case will correspond to having one of the k values in Equation 5. If r is the radius of a circle, the value $k = 1/r$ is often called the curvature. If r is infinite, the curvature is zero, and that makes sense: a straight line is not curved at all, and thus has zero curvature. Tiny circles are tightly curved and have a large curvature and vice-versa.

Now we can prove a very interesting result of Descartes' theorem: If we can find four mutually-tangent circles, each of which has a curvature that is an integer (possibly zero), then we can choose any three of them and construct an additional circle that is tangent to those three and different from the fourth which also has an integer-valued curvature.

Suppose we begin with four particular values of k_i that satisfy:

$$2(k_0^2 + k_1^2 + k_2^2 + k_3^2) = (k_0 + k_1 + k_2 + k_3)^2.$$

We can find an additional value for k_3 as follows. We simply need to solve for x in the following quadratic equation:

$$2(k_0^2 + k_1^2 + k_2^2 + x^2) = (k_0 + k_1 + k_2 + x)^2.$$

If we expand the equation above, we obtain:

$$x^2 - 2x(k_0 + k_1 + k_2) - 2(k_0k_1 + k_1k_2 + k_2k_0) + k_0^2 + k_1^2 + k_2^2.$$

The two roots of this equation add to $2(k_0 + k_1 + k_2)$, but we know that one of the roots is k_3, so the other is $2(k_0 + k_1 + k_2) - k_3$. Since all the k_i are integers, so must be the other root.

If we state this in terms of radii instead of curvatures, it simply means that if all the radii have the form $1/n$, where n is an integer, then the newly-generated circle will also have a radius of exactly the same form.

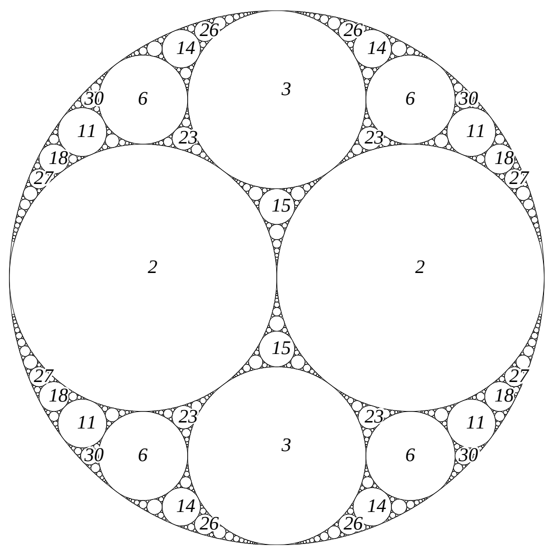

Figure 25: Tangent circles

Figure 25 shows what we can do with repeated inversions to construct more and more mutually-tangent circles. We begin with a large circle of radius 1 with two smaller circles inside having radii $1/2$ each, and two more having radius $1/3$ each. The numbers in some of the circles in the figure represent the inverse of the radius of that circle. The initial figure starts with five circles: the outer one and the four inside labeled with either a 2 or a 3. We can show that the combination of radii $\{1/1, 1/2, 1/2, 1/3\}$ satisfies Descartes' theorem, since:

$$2(1^2 + 2^2 + 2^2 + 3^2) = (1 + 2 + 2 + 3)^2.$$

In the equation above, replace one of the 2's on each side with an x and solve, and we obtain either $x = 2$ or $x = 6$, so the new mutually-tangent circle must have radius 6 (which means that the radius is $1/6$), and so on. Try checking a couple more.

Another application of this yields the so-called Farey circles. We begin with a straight line and a series of circles of radius $1/2$ tangent to it and to the next circle. Using Descartes' theorem (or in this case, just a simple application of the Pythagorean theorem), we can see that another circle fitting between any pair of circles of radius 1 and also tangent to the line will have radius $1/8$. If you use Descartes' theorem, remember that the line has curvature zero.

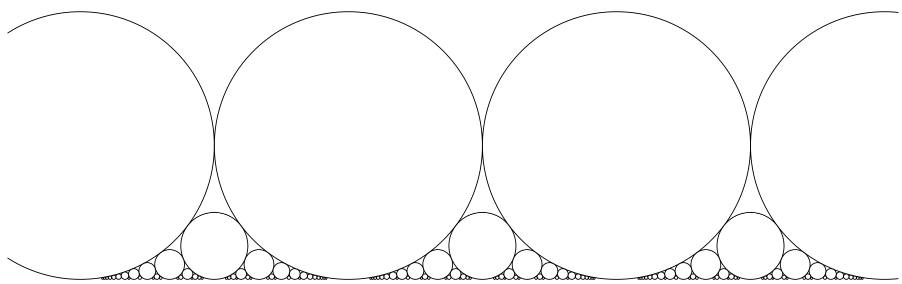

Figure 26: Farey (or Ford) circles

As above, we can repeatedly toss out one circle of a set (but always keep the line as one of the "circles") and we will generate the Farey circles. See Figure 26. These are also called Ford circles.

If we imagine all these circles placed on the plane it turns out that for any fraction p/q where p and q are relatively prime, then there is a circle centered at $(p/q, 1/(2q^2)$ and having radius $1/(2q^2)$. This is the complete set of the Farey or Ford circles.