

# Hackenbush

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## Abstract

This article describes the mathematics of the game Red-Blue Hackenbush, popularized by Conway, Berlekamp and Guy ([Elwyn R. Berlekamp, 1982], [Conway, 1976]). There are some notes included about how to use the game in a classroom.

## 1 The Rules

A Red-Blue Hackenbush position is a collection of points connected by line segments that are colored either red or blue. Some of the points are on a “ground line”. A segment is connected to the ground if at least one of its endpoints is on the ground, or if the segment is connected to another segment that is connected to the ground. In this article, the ground will always be a horizontal black line.

There are two players named “Red” and “Blue” who alternate moves. Red moves by cutting a red segment and Blue, by cutting a blue one. The cut segment is deleted together with any other segments that are no longer connected to the ground.

When a player is unable to move (in other words, when it is his turn and there are no longer any segments of his color), he loses. In fact, it’s useful to add the rule that there is a prize for winning, but there is an additional prize for every remaining segment of your color when the game ends. Thus it is better to win by a lot.

The mathematical name for a Hackenbush position is “graph”; in this case a graph with edges (line segments) colored red and blue. In fact, any connected graph with one special node (the ground) where all the edges are colored blue or red is equivalent to a Red-Blue Hackenbush game.

## 2 Software

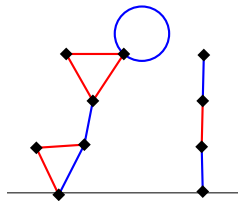
Greg Whitehead and Thane Plambek have written a great graphical implementation for the iPad. It’s called “HAKENBUSH” (no “C”) and it’s available for free at the Apple store.

I’ve written C code to play the game on arbitrary positions. It will play either side and will go first or second. There’s simple documentation inside the code file. Here’s the source:

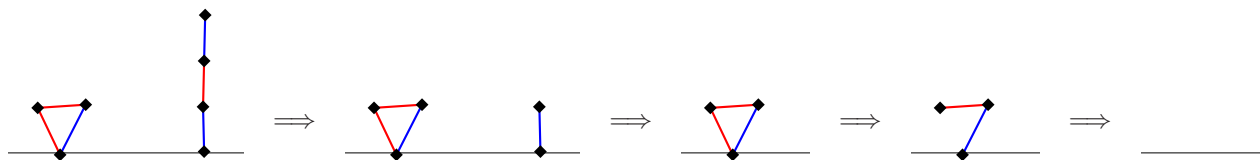
<http://www.geometer.org/puzzles/hack.c>

## 3 A Sample Game

A sample game appears below, and let us assume (now, and unless stated otherwise in this article) that Blue is to make the first move.



Blue's first move is to cut the blue segment that connects the two triangular structures on the left part of the diagram. The result is shown below at the left, and that cut will cause not only that segment to disappear, but also the triangle above it and the connected loop. It should be easy to follow the succeeding steps to the right as Red and Blue take turns hacking off segments:



When Blue cuts his final segment, the game ends and he wins (since Red has no more moves). Do you see why Red's last move was wrong and he could have won by cutting the red segment parallel to the ground? In addition, there are other non-optimal moves in this sample game. In fact, as we will be able to calculate later, this particular game, if played optimally by both players, can always be won by the second player. In other words if Blue moves first, Red wins, and if Red moves first, Blue wins.

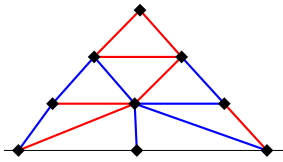
At this point, you may want to play with some simple games to try to learn what you can about Hackenbush before reading on. If you are a teacher, planning to use this in a classroom, one good way to start is to have the class work through the worksheet that appears as the final two pages of this document.

## 4 Simple Observations

- If the graph is empty (in other words, if there are no edges), then the player whose turn it is to move loses. Remember that if you cannot move, you lose. This may seem like a trivial observation, but one way to analyze Hackenbush positions is to consider moves that repeatedly simplify the situation, and if this method is applied recursively, a very convenient stopping point for the recursion is when the graph is empty.
- If some connected portion of the graph that is rooted to the ground contains only  $n$  edges of a single color then by cutting those edges in an appropriate order the player of that color can make  $n$  moves on that portion.
- If the red and blue edges do not interact – in other words, if cutting a red edge can never cause a blue one to be removed and vice-versa, then the result of a game played optimally with  $m$  such red edges and  $n$  such blue edges depends only on the value of  $n - m$ . If  $n - m$  is positive, Blue is certain to win. If  $n - m$  is negative, Red is certain to win. If it is zero, the second player wins. Why?
- Many Hackenbush positions can be broken up into components where the components have no edges or vertices in common except possibly for the vertex on the ground. If all the components are purely red or blue, we can assign a value of  $+n$  to a component with  $n$  edges if it is entirely blue and  $-n$  if it is entirely red. If we add these values for all the components in a position, then Blue, Red or the second player wins, depending on whether that value is positive, negative or zero.

Although the last item in the list above is a simple observation, the following is not: it turns out that we can assign numerical values to components where the edges do interact, and those values can be added to give the game a value. In this paper, we will always associate positive values with positions where Blue has an advantage and negative ones where the advantage lies with red. It will turn out in exactly the same way as it did for solidly-colored components that a position with value zero (with optimal play by both sides) will always yield a win for the second player.

Finally, notice that a game can look confusing, like the one below:

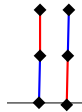


In this game there are no obvious red and blue components, although if you look at the game carefully, assuming that neither side makes a stupid move, there is no way for Red to cut off any blue edges and there is no way for Blue to cut off any red edges. Thus the value of this game is just the difference of the number of blue and red edges, and in this case, that number is 0.

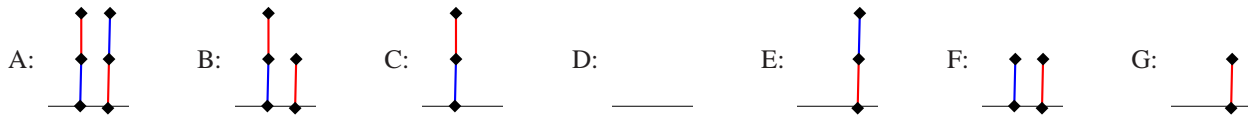
## 5 Analyzing Games

In any game that has to end with a winner or a loser (no ties) and where there are a finite number of possible moves and the game must end in finite time, there is (theoretically, at least) a method to determine the optimal strategy for the players.

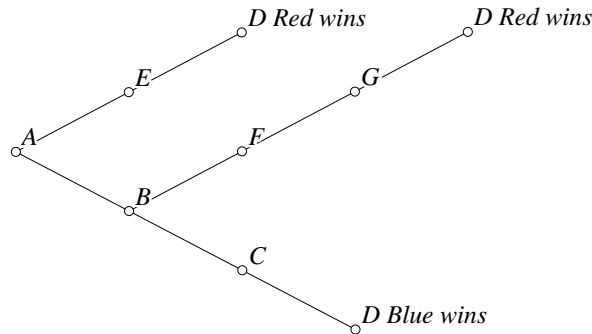
Let us analyze the following Hackenbush game under those assumptions:



In this section we assume that Blue makes the first move and that moves are alternated, so there are only 7 possible positions (A-G) that can be reached (including the initial position). Here they are:



We can now draw the following complete tree for the game:



To interpret the graph, begin at *A* on the left. Since it is Blue's move, he can choose to move to positions *B* or *E*, depending on which edge he cuts. The next move is Red's, et cetera.

But it's easy to analyze the situation backwards. Blue does not want to wind up at a node marked "Red wins," and vice-versa for Red. Thus if Red is in position *B*, he will move to *F* to guarantee a win. Since Blue makes the first move from position *A*, it is suicide to move to *E*, and although with optimal play by Red he is sure to lose anyway, at least if Blue

moves from  $A$  to  $B$  there is still a chance to wind up winning.

In a more complex tree, it is probably easier to begin at the rightmost nodes and work your way back, assigning either “Red wins” or “Blue wins” to each node, working back toward the beginning.

But in this case, by the analysis above, Red, if he plays correctly, can always win if Blue has the first move. Also, since the game is obviously symmetric in red and blue, if Red starts then Blue can always force a win.

## 6 More Observations

- If a blue edge is added to any position in any way, the position becomes better for Blue, and vice-versa.
- Symmetry is often an aid to analyzing a position.

The first observation above is pretty obvious, since you always want to have options to move, and having an extra option thus can't hurt. If you simply ignore the existence of the new edge, you still have all the moves in that position you had before, and possibly with luck, you can cut the new edge and get another move from the position.

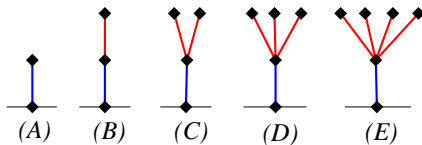
If you can divide a position into two symmetric parts where the structures are identical, but the colors reversed, then this is a position where, with good play, the second player can always win. Examples of symmetric positions appear in examples  $A$  and  $F$  in the previous section, and as examples  $G$  and  $I$  in the worksheet at the end of this document.

To see why the second person can always win, simply copy the move of the first person on the symmetric component, and since the two parts are identical except with swapped colors, there will always be a matching move. Thus there will always be an even number of moves, and when the moves run out, it will be the first player's turn, so he will lose.

In the next section, where we will begin to look at the problem of assigning a value to positions, where positions favoring Blue are assigned positive values and those favoring Red, negative ones, we can see that if two positions are exactly the same except that all of the colors have been swapped, then the value of one will be the negative of the value of the other.

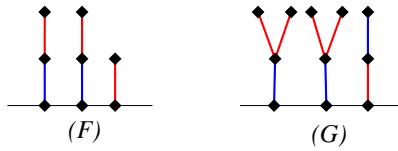
## 7 Some Fractional Games

In this section we will make some convincing arguments for why certain fractional values can be assigned to certain positions. We will start with the idea that components of positions that are made up of entirely  $n$  blue segments will have a value of  $+n$ , and if there are  $n$  red segments, the value will be  $-n$ . Consider the following components:



Component  $A$  is straightforward: Blue has exactly one move, and this component should be assigned a value of  $+1$ . In all the other positions, Blue can win whether he starts or not, but each successive position to the right is less desirable for Blue since Red has more and more options. In a position that contains a component like  $E$  above, if Blue needs to make moves in other components because they are even more critical, Red has up to four “free” moves.

So if the value of component  $A$  is  $+1$ , what values should we assign to components  $B$ ,  $C$ ,  $D$ , and  $E$ ? To find out, consider the complete position labeled  $F$  below:



It is worthwhile to spend enough time with this game to see that it has value 0: in other words, with optimal play by both sides, whoever plays first will lose. That means that somehow the two copies of position  $B$  above are equivalent to a single disjoint red segment (with value  $-1$ ). This means that it might be reasonable to assign a value of  $+1/2$  to component  $B$ .

Now consider position  $G$ . The rightmost component of  $G$  is the same as  $B$ , but with the colors reversed. That means that its value should probably be  $-1/2$ . Again (and this is a little more difficult) you can analyze position  $G$  and see that it can always be won by the second player with optimal play by both sides, so the total game has value 0. This would mean, of course, that position  $C$  above should probably be assigned the value  $+1/4$ .

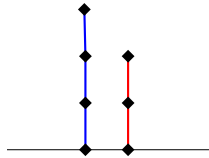
**Note:** It is not too hard to prove that these games are winnable by the second player. Think about what we said in Section 6 about symmetry.

**Another Note:** If you were shown a position that somebody claimed had a value of  $-3/4$  what would make a convincing argument that this is true (or false)?

It will also turn out that reasonable values for  $D$ ,  $E$ , et cetera ought to be  $+1/8$ ,  $+1/16$ , et cetera. More on this later.

## 8 Finding a Game's Value

Let's consider a very simple game:



There are two independent components that are completely blue or completely red. A totally methodical way to analyze the game would be to trace the results of every possible combination of moves. For a game this simple, the best moves by each side are obvious: hack off the uppermost segment each time. Although it would be more inefficient, it wouldn't hurt to look at every possible move.

According to our previous discussions, a reasonable value for this game would be:  $3 - 2 = +1$  (three blue moves for  $+3$  and two red moves for  $-2$  when added yield  $+1$ ).

If Blue has to move from this position, the resulting positions have values of  $0$ ,  $-1$  and  $-2$ . If instead it is Red's move, the resulting games would have values of  $+2$  and  $+3$ . Remember that when Red makes a cut, the situation is always worse for him, since he now has fewer options, so the game values increase (become better for Blue). Similarly, any Blue move makes the situation worse for him (and better for Red), so the game value decreases (which is what Red wants).

The true value of the game must therefore be larger than all the game values after Blue moves, but smaller than all the game values after Red moves. In fact, suppose the game values after a Blue move are  $B_1, B_2, \dots, B_n$  (assuming Blue has  $n$  possible moves) and the values after a Red move are  $R_1, R_2, \dots, R_m$  (assuming that Red has  $m$  possible moves). The true game value  $V$  must be larger than all the  $B_i$  and smaller than all of the  $R_i$ . We will indicate this as follows:

$$V = \{B_1, B_2, \dots, B_n | R_1, R_2, \dots, R_m\}.$$

(This doesn't tell us how to calculate  $V$ ; it simply is how we will write that value in terms of the  $B_i$  and  $R_i$ .)

For the sample game above, there are three possible Blue moves and two possible Red moves with the values stated above, and we can write the value of the game as:

$$\{-2, -1, 0|2, 3\}.$$

The game value will be a number that lies above all the Blue-move-created values on the left and below all the Red-move-created values on the right. We can ignore the “bad” moves and all that really concerns us are the largest value on the left and the smallest value on the right:

$$\{-2, -1, 0|2, 3\} = \{0|2\}.$$

In fact, the value of the game is the “simplest” number that lies between the largest number on the left and the smallest number on the right. In the case of the game we’re considering, we need the simplest number between 0 and +2, which is “obviously” +1.

Let’s look at one more example: the game labeled (*B*) in Section 7. We decided in that section that a reasonable value might be +1/2, since combining two of them with a −1 game seemed to make a zero game where the second player could win with an optimal strategy, but let’s try to see what the “simplest rule” would do for us. In that game, there is only one move available to each player. If Blue moves, everything is chopped away, yielding a position with value 0. Red’s only move leaves a single blue segment yielding a position with value 1. The value of that game ( $\{0|1\}$ ) should thus be the simplest number between 0 and 1, and +1/2 is not a bad candidate.

Finally, we have to be able to calculate values of games where one of the players has no moves. For example, suppose the position has exactly two blue segments, so Red has no moves, but Blue can create a position with values of +1 or 0. Our notation for the value of this game would be:

$$\{0, +1| \},$$

and the best way to interpret this is as “the simplest number larger than +1, which would be +2. In the opposite game where there are initially two red and no blue segments, Blue has no moves, but Red can move to 0 or −1. Similarly, we’d be looking for the simplest number smaller than −1, and a good case can be made for −2.

Finally, the value of the empty game is  $\{ | \}$  (no moves available for either side), and so we’re seeking the “simplest number”, for which zero is a pretty reasonable choice.

With these observations, the value of *any* Hackenbush position can be calculated, as long as you’ve got a good idea of what is meant by the “simplest” number.

## 9 Surreal Numbers

**Note:** In this tiny section there is no time to completely prove even the most basic theorems about the surreal numbers. It will turn out, for example, that there are an infinite number of ways to represent each one, but that every one of those ways will behave in a reasonable way when calculations are made with it. This should not be surprising: the same thing happens in classic rational numbers:  $1/2 = 2/4 = 3/6 = 4/8 = \dots$ , for example. Another missing theorem we will have is that when a surreal number corresponds to a standard rational number (and all of them will, here), then the surreal number sum is the same as the rational number sum, and the game represented by the combination of two components having values  $V_1$  and  $V_2$  will have a value of  $V_1 + V_2$ .

The search for simple numbers will use a (tiny) subset of the so-called “Surreal Numbers” from Conway ([Conway, 1976]) and popularized by Knuth ([Knuth, 1974]) in his book of the same name. The idea is this: imagine that we start with nothing at generation zero, and in each generation, we produce all possible divisions of the numbers we already have, presented in the game notation, where all the numbers on the left are smaller than all the numbers on the right.

In generation zero, all we have is this:

$$\{ | \},$$

which we will call 0.

In generation 1, we now have these additional possibilities:

$$\{ |0\} \text{ and } \{0|\}.$$

We thus seek the simplest number smaller than 0 and the simplest number larger than zero, so in generation 1, we add the numbers  $-1$  and  $+1$ .

For this article, we are only going to work with finite sets of numbers, so when we seek new numbers we only need to look for them between numbers we already have. Thus in generation 2, we'll have to add these:

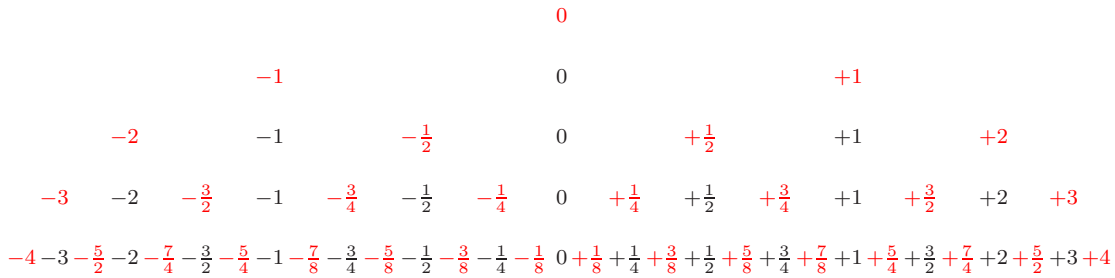
$$\{ |-1\}, \{-1|0\}, \{0|1\}, \text{ and } \{1|\}.$$

These will be the numbers:  $-2$ ,  $-1/2$ ,  $+1/2$  and  $+2$ .

The next generation will yield:

$$-3, -3/2, -3/4, -1/4, +1/4, +3/4, +3/2 \text{ and } +3.$$

The number  $-3/4$  is the simplest one between  $-1$  and  $-1/2$ , and so on.



The top line in the illustration above shows all the numbers that are created in generation 0. The next line, includes all the generation 0 numbers, but the newly-introduced (generation 1) numbers are drawn in red. The next line shows all the numbers that exist in generation 2, but again, with the newly-introduced numbers in red, and so on.

Among a set of numbers, we define the simplest one to be the number which appeared in the earliest generation. Thus, for example, the simplest number between  $+1/8$  and  $+3/4$  is  $+1/2$ . Similarly, the simplest number between any negative number and any positive number is 0 (zero). When you are looking for the simplest number, remember that it may have occurred much earlier in the generations than the numbers you are looking at. For example, if you have the number  $0 | 3/4$ , its value is  $1/2$ . The knee-jerk reaction is to take the average of 0 and  $3/4$ , or  $3/8$ , which is wrong.

Note that if we let this process go on forever, we will eventually obtain all rational numbers, positive and negative, that have a power of 2 in the denominator. For the Red-Blue Hackenbush game, these are all the numbers that are required.

Here are a couple of examples of the fact mentioned in the first paragraph of this section; namely, wildly-different representations of the same number:

$$\begin{aligned} 0 &= \{ |\} = \{-17/64|47/128\} = \{-1/1024|1000000\} = \dots \\ 1 &= \{0|\} = \{1/2|9/8\} = \{1/16|1000000\} = \dots \\ 1/8 &= \{0|1/4\} = \{1/64|3/8\} = \dots \end{aligned}$$

### 9.1 The rest of the surreal numbers

This subsection is not important to learning about Hackenbush, but the true surreal numbers are fascinating. Here are a couple more comments about them.

The true surreal numbers are constructed by continuing on after “going on forever.” The “next” generation after that will give all the real numbers plus a sort of positive and negative infinity. For example,  $\pi$  will be generated by this number ( $\mathbb{Z}$  is the set of integers):

$$\{m/2^n : m, n \in \mathbb{Z} \wedge m/n < \pi \mid m/2^n : m, n \in \mathbb{Z} \wedge m/n > \pi\},$$

and the infinities (called  $\omega$  and  $-\omega$ ) are generated by:

$$\begin{aligned} \omega &= \{m : m \in \mathbb{Z} \mid \} \\ -\omega &= \{ \mid m : m \in \mathbb{Z} \} \end{aligned}$$

In other words,  $\omega$  is the “simplest” number larger than all the integers, and  $-\omega$  is the simplest one smaller than all of them. We can also obtain in this step infinitely small numbers, like:

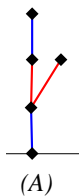
$$1/\omega = \{0 \mid 1/n : n > 0 \wedge n \in \mathbb{Z}\},$$

(in other words the number smaller than all positive fractions, but bigger than zero).

In the next generation, we’ll obtain stuff like:  $\omega + 1$ ,  $\omega - 1$ ,  $1/2\omega$ ,  $\pi + 1/\omega$ , et cetera.

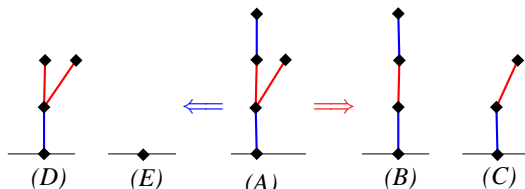
## 10 Calculating a Game Value

In this section, although it will be, perhaps, a bit boring, we will show how the rule in the previous sections allow us to work out the value of any Hackenbush game. We will work with this example:

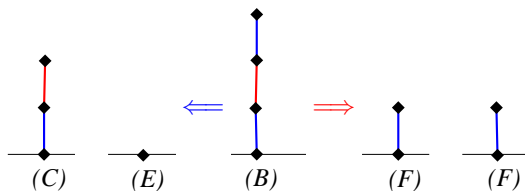


(A)

Beginning with this game, the pair of games on the left show all the possibilities of positions that can be obtained with a Blue move, and all the ones on the right are the possibilities after a Red move.



From previous work, we know the game values of all the components above except for the tower or alternating colors (B), and a similar listing of all the moves possible from that are (identical games are assigned the same label, so all the empty games, for example, are called (E)):



We know that the values of (C), (D), (E) and (F) are, respectively,  $+1/2$ ,  $+1/4$ , 0 and  $+1$ , so we can find the value of B:

$$(B) = \{+1/2, 0 \mid +1, +1\},$$



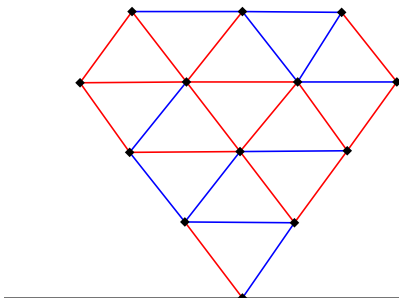
and the simplest number in that range is  $+3/4$ .

Now that we know the value of  $(B)$ , we can obtain the value of  $(A)$  as follows:

$$(A) = \{+1/4, 0 | +3/4, +1/2\},$$

which will yield:  $(A) = +3/8$ .

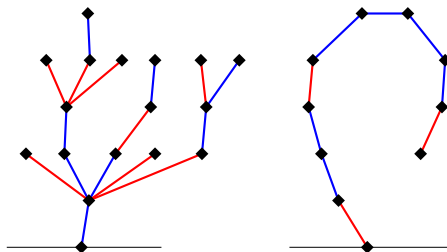
While this method for calculating a game value is conceptually simple, the number of sub-games that need to be analyzed can be enormous. For example, if the following game is completely analyzed, 31,168,746 positions occur:



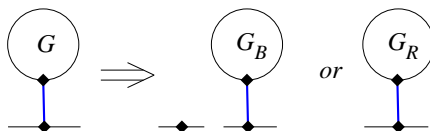
## 11 Stalks, Trees and Loops

### 11.1 Stalks and Trees

A “tree” is a connected position where there is a unique path from any node to the “ground” node. In other words, in a tree, there are no loops. A “stalk” is a very special kind of tree where there are no branches: every node has at most two segments connected to it. In both a tree and a stalk, nodes can be colored arbitrarily. In the figure below, the position on the left is a tree and the position on the right is a stalk (and also a tree, of course).



It turns out that there are some nice recursive methods to calculate the value of a tree or a stalk, and those recursive methods can sometimes be used to help calculate the values of more complex positions. The idea is based on the following theorem based on the figure below:



If  $G$  is some Red-Blue Hackenbush position having value  $x$  (for example, the position on the far left of the figure above), then if the grounded segments of  $G$  are connected instead to a single segment of color blue, then the value of “ $G$  on a stick” is equal to the first value from the series:

$$\frac{x+1}{1}, \frac{x+2}{2}, \frac{x+3}{4}, \frac{x+4}{8}, \frac{x+5}{16}, \frac{x+6}{32}, \dots$$

for which the numerator of the expression exceeds 1.

(If the “stick” under the game  $G$  with value  $x$  is instead red, then the value of that game on a stick is given by the first term in the series:

$$\frac{x-1}{1}, \frac{x-2}{2}, \frac{x-3}{4}, \frac{x-4}{8}, \frac{x-5}{16}, \frac{x-6}{32}, \dots$$

for which the numerator of the expression is exceeded by  $-1$ .)

These formulas can be obtained by looking at the results of Blue and Red moves. We’ll do that for the situation above where we have a game on top of a blue segment as in the figure above. In that figure, Blue can either cut the stick supporting  $G$ , in which case the game becomes empty (and has value zero), or Blue can make his best move inside the game  $G$ , generating a game  $G_B$  on a blue stick. If Blue has a valid move in  $G$ , that’s always better to do than to cut the lowest segment since the result will be something on top of a blue segment whose value is certainly larger than zero. So Blue should never cut the bottom segment if there are any blue segments available to cut above it. If it’s Red’s move, his best bet is to make the best move in  $G$ , yielding the position  $G_R$  on a stick.

If the value of  $G$  is  $x$ , we actually need to construct the function mapping  $x$  to the value of a game with  $G$  on a blue stick. Let’s call that function  $f(x)$ . Note that we can always write  $x$  as  $\{x_B \mid x_R\}$ , where  $x_B$  and  $x_R$  are simpler numbers than  $x$  (or are empty).

What is  $f(0)$ ? That’s equivalent to a null game on a (blue) stick, or 1, so we have  $f(0) = 1$ . Now look at the values of  $f(x)$  where  $x$  is in the next generation; namely,  $x = -1$  or  $x = 1$ . If  $x = 1$  it’s equivalent to two blue segments, or  $f(1) = 2$ . We can continue with positive values, and clearly,  $f(n) = n + 1$ , assuming that  $n \geq 0$ .

How about when  $x = -1$ ? Well, a typical game with value  $-1$  could come from  $\{ \mid 0 \}$ , where Blue has no moves and Red can make a game of value 0. 0 on a blue stick has value 1, but if Blue has to cut the stick, he’ll obtain a game of value 0, so the value  $f(-1) = \{0 \mid 1\} = 1/2$ .

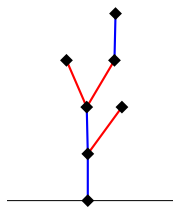
What is  $f(-2)$ ? This could come from the game  $\{ \mid -1 \}$ . Again, Blue’s only move is to cut the bottom stalk, but Red can make a game of value  $-1$  on a blue stick, and we know (from the previous paragraph) the value of that game is  $1/2$ . Thus  $f(-2) = \{0 \mid 1/2\} = 1/4$ .

The same pattern will continue for  $f(-3)$ ,  $f(-4)$ ,  $\dots$ , and we will obtain the following table of values for  $f$ :

$x =$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x) =$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5	6

The other values for  $f(x)$  can be linearly interpolated from the list above, yielding the formulas stated earlier.

Let’s now use this theorem to calculate the value of the position below:



The topmost single blue segment has value 1. The node below that has two red segments. One is a single segment with value  $-1$ , but the one on the right is blue on red (which should have value  $-1/2$ ). To see that the  $-1/2$  agrees with the

formulas, we need to look at the numerators:  $x - 1, x - 2, \dots$ , where  $x = 1$  (the value of the single blue strut) and find the first one exceeded by  $-1$ . We see that  $1 - 1$  doesn't work, nor does  $1 - 2$  (which is equal to  $-1$ , but is not exceeded it), but finally,  $1 - 3 = -2$  is smaller than  $-1$  therefore the stalk with blue on top of red is  $\frac{1-3}{4} = -1/2$ .

The next node down, which is one step above the ground, has a single red strut with value  $-1$ , plus a blue strut with a position equal to  $-3/2 = -1 + (-1/2)$  on top of it. Now  $x$  will be  $-3/2$  and we need to look at  $x + 1, x + 2, \dots$  until one is larger than 1. We see that  $x + 3$  works, so the value of the position is  $(+3/2)/4 = +3/8$ . That, plus the red segment with value  $-1$  yields a game of value  $-5/8$  on top of a blue stick. We have to add 2 to that to exceed 1, so the value of the full game will be:  $((-5/8) + 2)/2 = 11/16$ .

When we evaluate trees by this method (as above), to find the value of the game above each node we need to find the values of the trees above the node and add them. Then apply the rule to find that value of that node. We can work our way to the bottom of the entire tree in that way.

Note that the formula even works for a single blue (or red) segment. The game above it is empty, so has value zero. Since  $0 + 2$  exceeds 1, the value of the single blue segment will be  $2/2 = 1$ , which is correct.

To make sure you understand the rule, try it out on some of the stalks in the fourth row of examples in Section 12.

In fact, assuming that the base segment of the stalk is blue (for red, just take the negatives of these numbers), there is a simple algorithm to calculate the value of a stalk:

- Count the number of blue segments that are connected to the in one continuous path. If there are  $n$  of them, start with the number  $n$ .
- For each new segment going up, assign the value of that segment to be half of the one below it, and add it to the sum if it is blue, and subtract if it is red.
- When you reach the top of the stalk, that's your final value.

For example, consider the stalk whose segments are, starting from the ground: *BBBRRBRRBR*. We begin with 3 because of the three blue segments. The next red adds  $-1/2$ , the next red adds  $-1/4$ , the following blue adds  $+1/8$ , and so on. Thus the value for this stalk will be:

$$3 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} = 2\frac{37}{128}.$$

With this information, it is not too hard to construct a stalk with any particular value you wish whose denominator is a power of 2. (You might want to think about how to do this before reading the following paragraphs.)

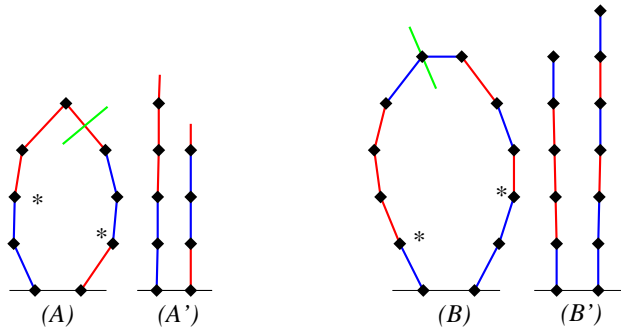
Rather than write down the algorithm, let's just look at a complete example. Suppose you wish to construct a stalk having a value of  $3\frac{21}{32}$ . This is larger than 3, so begin with a stack of three blue segments, so the value of the stalk so far is 3. Your target is larger than 3, so add another blue, making the value 4: too large. Add a red, reducing 4 by  $1/2$  to  $3\frac{1}{2} = 3\frac{16}{32}$ : too small. Add a blue, increasing the value by  $1/4$  to  $3\frac{3}{4} = 3\frac{24}{32}$ : too large. Add a red, reducing the value by  $1/8$  to  $3\frac{5}{8} = 3\frac{20}{32}$ : too small. Add a blue, adding  $1/16$ , and finally a red, subtracting  $1/32$  to achieve the final value. Thus the required stalk, starting from the ground, will be: *BBBBRBRBR*.

This has interesting applications if you are trying to make a Hackenbush position that is difficult to win, but not impossible. Notice that positions with large values are generally easier for Blue to win, so if you've got a position with value, say, 3.5, in a sense Blue can make three errors that give away a whole segment and can still win the game. But a game with value  $1/64$  will allow almost no wrong moves, assuming that Red plays well.

So to make a difficult (but winnable) position for Blue, first make a confusing (with loops and stuff) position and find its value  $v$ . Next, build a stalk whose value is  $1/64 - v$  and add that stalk to the position. You now have a position worth  $v + 1/64 - v = 1/64$ .

## 11.2 Loops

A "loop" is like a stalk with both ends connected to the ground, or connected to a single node somewhere within a more complex position. In the figure below, the positions labeled (A) and (B) are loops.



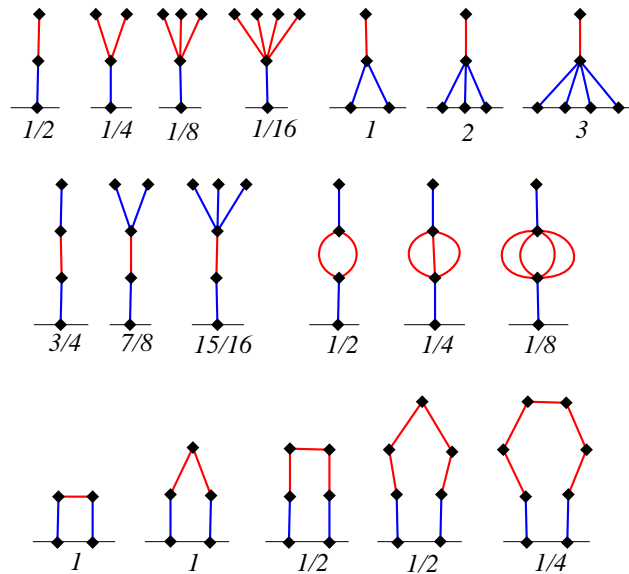
Berlekamp has proved that to find the value of a loop by finding the points nearest the two connections to the ground where the segments change color. In the illustration above, these points are marked with asterisks (\*). Next, divide the rest of the loop into two equal pieces. If there are an even number of segments (as in (B)), then cut at the node. If there are an odd number, cut the middle segment in half and treat it as two segments (as in (A)). The value of the loop is the sum of the values of the two segments; in other words, the value of (A) is the same as the value of (A') and the value of (B) is the same as the value of (B').

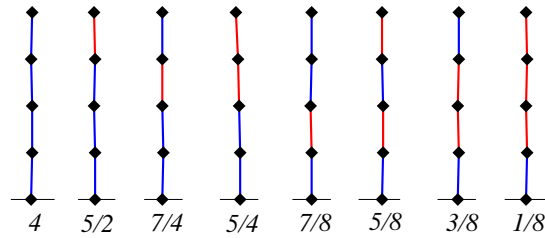
It turns out that (A) is worth  $3/4$  and the two stalks in (A') are worth  $9/8$  and  $-3/8$ , and  $9/8 - 3/8 = 6/8 = 3/4$ .

for (B), the full loop has value  $15/8$  and the two stalks in (B') have values  $3/16$  and  $27/16$ . We can check:  $3/16 + 27/16 = 30/16 = 15/8$ .

## 12 Sample Position Values

This section will just list the values of some relatively simply Hackenbush positions. Many of them are arranged in series so you can see a pattern in the values. Remember that if you swap the red and black segments, the value of the resulting position is the negative of the original position.





## 13 Other Versions of Hackenbush

We will not have the space to cover them here, but there are other versions of the game which are analyzed in Conway’s books ([Elwyn R. Berlekamp, 1982], [Conway, 1976]).

One possibility is simply to allow infinitely many red and blue segments. These games will have values that correspond to more of the surreal numbers and they are not wildly more complicated. Here’s an interesting example. Suppose there are two infinitely-long stalks, one all of red segments and the other all blue. Again, the second player always wins, because as soon as the first player moves, he has to cut somewhere in that infinite stalk, and that will leave him with just a finite number of his segments. The other player simply has to cut higher to guarantee a win, and can cut as much higher as he pleases, so his advantage after the second move can be arbitrarily large.

Another possibility is to introduce green edges that may be cut by either player. Games may have only green edges, or they may have a combination of green, red and blue edges. Finally, these games that include some green may also have an infinite number of segments.

### 13.1 Analyzing Some Infinite Games

If we look at a stalk entirely composed of segments of the same color, it is clear that the game value of the stalk is just the height of the stalk or its negative. In other words, a stalk of 20 blues is worth 20, a stalk of 1000 red segments is worth  $-1000$ , et cetera.

What if we consider a stalk of blue segments that never ends? It’s hooked to the ground but goes up forever. It’s clear that the value of this stalk is larger than the values of any finite stalk of blues, so its value, in some sense, should be  $\infty$ . Using the surreal representation, the value of the stalk should be:

$$\{ |1, 2, 3, 4, \dots \} = \infty.$$

We can test this assumption by noticing that if you are Blue and have this as one of the components in a game, you can beat Red if Red has any position with a finite negative value, whether it be  $-1$ ,  $-1000$ , or  $-10000000000$ . When it is your move, you simply cut high enough that the remaining (now finite) stalk has a large enough positive value to beat Red’s number.

One thing that is interesting about these infinite stalks is that although the “game board” is infinite, any game played on it is finite. There is no limit to how long such a finite game could be, by as soon as you commit to the first cut on that stalk, there are only a finite number of moves left. The same is true if you have two (or ten or a million) such stalks. Each time you make the initial cut on a stalk, there are only a finite number of moves available on that stalk, and when you’re finally done making them, you have to commit on the next stalk to a cut which makes it finite, et cetera.

As we expect, The game where each color has an infinite stalk has value zero, since whoever makes the first move commits to a finite number and the opponent can match that and win by symmetry, or, in fact, cut higher and finish with as large an advantage as she’d like.

It’s also interesting to consider “ $\infty + 1$ ”: an infinite stalk plus a single additional stalk of length 1. This is clearly larger than  $\infty$ , but not as large as  $\infty + 2$ . You can also look at games with multiple stalks, representing surreal numbers like  $\infty \times n$ , where  $n$  is the number of infinite blue stalks you have.

Another easy-to-analyze situation occurs when we consider finite stalks that contain a single blue segment at the bottom, and a longer and longer sequence of red segments above it. Starting with zero red stalks, the successive values are:  $1, 1/2, 1/4, 1/8, 1/16, \dots$ . So if there are an infinite number of red segments, the value ought to be:

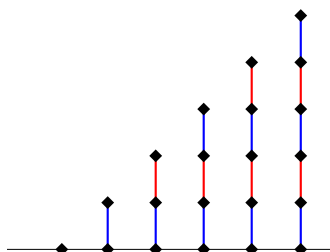
$$\{ |1, 1/2, 1/4, 1/8, \dots \}.$$

This “number” has to be smaller than any positive number, but larger than zero, so it has a value that we can think of as  $1/\infty$ . Again, it’s easy to see that this is reasonable. No matter how many of these Blue has, he can’t win in any game where the rest of the components add up to any negative value for Red. Try playing 1000 of them against a single red stalk with one segment, and Red clearly wins, whether he starts or not.

It’s also easy to see why two such stalks for Blue would beat a single such stalk for Red, so we have (in surreal numbers and in Hackenbush games):

$$0 < 1/\infty < 2/\infty < 3/\infty < \dots < 1/8 < 1/4 < 1/2 < 1.$$

Let’s look at another, more complex, infinite game. Consider the sequence of finite stalks below, beginning with the empty stalk:



The easiest way to work out the stalk values is to use the methods in Section 11. Start with a red over blue example yielding  $+1/2$ . Then notice that the inverse, blue over red, would have value  $-1/2$  and work out the value of that over blue, yielding  $3/4$ . The inverse of that will be  $-3/4$ , so put that over blue to obtain  $+5/8$ , and so on.

If we continue in this way to work out the values for these stalks from left to right, we obtain:

$$0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \frac{43}{64}, \frac{85}{128}, \frac{171}{256}, \frac{341}{512}, \dots$$

If we convert these values to decimal approximations, here’s what we get:

$$0, 1, .5, .75, .625, .6875, .6562, .6719, .6641, .6679, .6660, \dots$$

It sure looks like we’re approaching  $2/3$ , doesn’t it. Here is a pattern that will allow us to do the exact calculation. Beginning from the first term (with value 0), look at the changes in value of the fractions from one to the next. They are:

$$+1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, +\frac{1}{16}, -\frac{1}{32}, +\frac{1}{64}, \dots$$

If we add all of these together (it’s just a geometric series with initial value 1 and ratio  $-1/2$ , we obtain (for the infinite sum)  $1/(1 - (-1/2)) = 2/3$ . Thus, a “reasonable” value to assign to an infinite stalk with a blue segment on the bottom and alternating colors going up forever would be  $2/3$ .

One way to test this, of course, would be to see if the game consisting of three of these stalks together with two single red segments is a game having value 0: in other words, the second player in such a game always wins, with optimal play.

The fact that the stalks are infinite doesn’t cause any real problems. Once a player takes a hack at a stalk, almost all of the segments disappear and the remainder is finite.

Try to think about how to play this game before reading on.

Notice that any finite piece of the stalks above with a blue segment on the top will have a value larger than  $2/3$  and any finite stalk with a red segment on top will have a value less than  $2/3$ . Also, the taller the stalk is, the closer to  $2/3$  is the value.

Assume first that Red has the first move. If he cuts one of the single red stalks, the game value will be approximately  $-1 + 3(2/3) = 2$ , so that's probably a bad idea. So Red should cut one of the infinite stalks, but where? The higher he cuts, the smaller the value, which is what he wants, since large values are good for Blue. But he has to commit to some cut, at which point Blue simply cuts much higher on one of the remaining stalks, so that the sum of those two finite stalks is greater than  $2/3 + 2/3$ . If Red cuts the third stalk, the value of the remainder will also be larger than  $2/3$  and we have a finite game with the three remaining parts of the infinite stalks being worth more than 2, so Blue wins. But if Red cuts some other place, Blue simply chooses a cut high enough to make the sum of the three stalks larger than 2, and again, he's guaranteed a win.

Thus if Red starts, then Blue wins.

Now assume that Blue needs to go first. We can use similar reasoning in this case. Blue wants to cut one of the infinite stalks as high as possible, but however high he cuts it, Red can cut one of the other stalks higher than that, making the value of the game negative, et cetera.

## References

- [Conway, 1976] Conway, J. H. (1976). *On Numbers and Games*. Academic Press, London, New York, San Francisco.
- [Elwyn R. Berlekamp, 1982] Elwyn R. Berlekamp, John H. Conway, R. K. G. (1982). *Winning Ways, Volume 1: Games in General*. Academic Press, London, New York, Paris, San Diego, San Francisco, São Paulo, Sydney, Tokyo, Toronto.
- [Knuth, 1974] Knuth, D. E. (1974). *Surreal Numbers*. Addison-Wesley, Reading, Massachusetts, Menlo Park, California, London, Amsterdam, Don Mills, Ontario, Sydney.

# Hackenbush Worksheet

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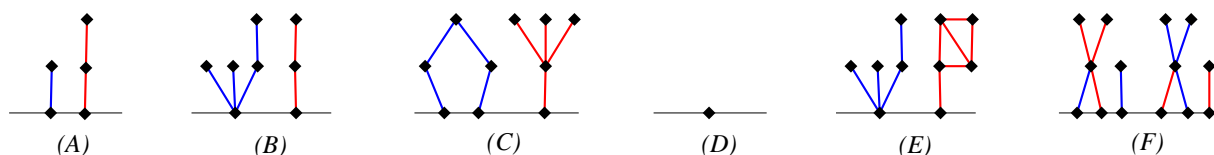
<http://www.geometer.org/mathcircles>

December 15, 2011

A Red-Blue Hackenbush position is a collection of points connected by line segments that are colored either red or blue. Some of the points are on a “ground line”. A segment is connected to the ground if at least one of its endpoints is on the ground, or if the segment is connected to another segment that is connected to the ground.

There are two players named “Red” and “Blue” who alternate moves. Red moves by cutting a red segment and Blue, by cutting a blue one. The cut segment is deleted together with any other segments that are no longer connected to the ground.

When a player is unable to move (in other words, when it is his turn and there are no longer any segments of his color), he loses. In fact, it’s useful to add the rule that there is a prize for winning, but there is an additional prize for every remaining segment of your color when the game ends. Thus it is better to win by a lot.



For example, in figure (A), if it is Blue’s turn, he will have to cut the only blue segment. It is now Red’s move and he can cut either the lower or upper segment. If he cuts the lower one, both disappear, since the upper one will no longer be connected to the ground, but if he cuts the upper one, a single red segment will remain, connected to the ground. In either case, it is now Blue’s move, and since there are no blue segments, the game is over and Blue loses. If Red is trying to win *and* have remaining red segments, he will obviously cut the upper segment, leaving one at the end of the game. If Red has to start the game, he will win if he starts by cutting the upper segment and he will lose if he cuts the lower one on his first move. Obviously, his first move should be to cut the upper one.

For each of the figures (A) through (F) below, figure out which player wins (assuming optimal play), and by how much, assuming that Blue moves first and also if Red moves first. For figure (A), if Blue starts, Red wins and at the end there is one red segment left: (+1). If Red starts and plays well, Red also wins, but with zero red segments left (0). Do the same analysis for all the figures and fill in the rest of the chart.

Figure	Blue Starts	Red Starts
(A)	Red Wins (+1)	Red Wins (0)
(B)		
(C)		
(D)		
(E)		
(F)		

Notice in all the examples above, the red and blue segments are independent, in the sense that a Blue cut can never remove red segments and vice-versa.

- In situations like those above, if there are  $m$  red segments and  $n$  blue segments, what would the entry in the table look like (in other words, who wins, and by how much)?
- If Blue has  $n$  blue segments all of which are connected to the ground via blue segments, can he always cut them off in an appropriate order so that he can make a total of  $n$  moves?



- Game ( $D$ ) (called the “null game”) is still a game. You just don’t want to be the first one to move!
- In game ( $F$ ), no Blue move can cut a red segment and vice-versa. What does this mean?

Now try the same analysis for the following three games:

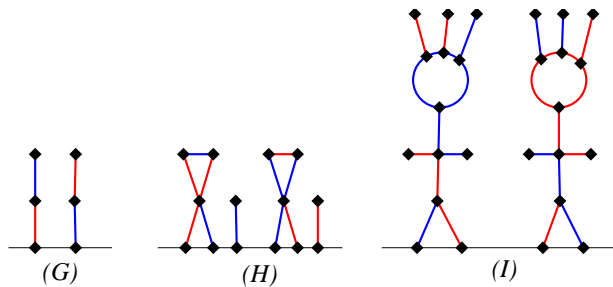


Figure	Blue Starts	Red Starts
(G)		
(H)		
(I)		

- How can symmetry be used to analyze Hackenbush positions?

A good way to think about Hackenbush positions is to try to attach numbers to them as we saw in the first set of exercises. Basically, when the colored segments do not interfere with each other, you can work out which side has the advantage by just counting the number of red and blue segments and considering the difference. In situations like this, let each blue segment be worth  $+1$  and each red segment  $-1$  and add them together. If there are more red segments, the total will be negative, and if there are the same number, the total will be zero.

In situations like the first 6, a positive total means Blue will always win. If the total is negative, Red always wins. If the total is zero, the first player loses.

What’s amazing is that *all* Hackenbush games can be assigned a number. Analyze the three positions below:

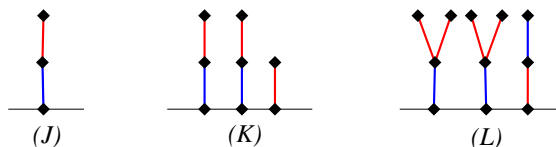


Figure	Blue Starts	Red Starts
(J)		
(K)		
(L)		

- In position ( $J$ ), Blue clearly has an advantage, but it’s not as big as it would be without the red segment. So if we are to assign a number to it, that number should be larger than 0 and less than 1.
- Position ( $K$ ) is always a win for the second player, so the total position value ought to be 0. Since it includes two copies of position ( $J$ ) and one independent red segment (worth  $-1$ ), it’s reasonable to assign the value of  $1/2$  to position ( $J$ ).
- Can you see why position ( $L$ ) provides evidence that the “Y”-shaped components on the left ought to be assigned a value of  $1/4$ ?