Geometry and Geography

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1 Pedagogical Advice

I have been leading mathematical circles using this topic for many years, and it seems to work well for both young and old students. The youngest group I've tried it on is third graders, and it works equally well for high school students. Of course the selection of questions and the presentation varies, depending on the level of the students. I usually start the circle with a general plan, but depending on the direction of the discussion, I omit some topics or go off on a tangent if that seems right. There is plenty of material here for two hours, so in most cases you will never cover all the topics.

A good mathematician must be good at visualization. Almost all problems have a geometric interpretation, and being able to visualize and manipulate geometric images in your head can vastly improve your ability to solve mathematical problems. Visualizing things in two dimensions is difficult, and three-dimensional visualization is very difficult. One of the main goals for presenting this circle is to improve the students' three-dimensional visualization ability.

It's probably easier for students to visualize the Earth as a ball in space than to visualize a mathematical sphere. They also know what the moon, sun and stars look like. Many have seen models of the solar system with planets orbiting the sun and moons orbiting some planets. For all these reasons, geography is more concrete and easier to visualize than spheres or circles or planes in 3-space.

Most students have looked at models of the Earth and know about the equator and lines of longitude (which are great circles or parts of them) have seen that most globes are mounted on a tilted axis, and that there are the perhaps somewhat mysterious arctic and antarctic circles together with the tropics of Cancer and Capricorn. Many know that when it is summer in the United States, it's winter in Brazil, but they don't know why. Or they know that the moon has phases, but have never really thought about why. A globe of the earth, a ball or two to represent the sun and/or moon, and a flashlight can make a lot of this obvious. Each of these can serve as a starting point for discussions.

The material that follows is arranged in two ways. The following section simply contains the questions arranged in a reasonable order for presentation. You could duplicate these pages as a handout if you'd like the students to have a copy during the discussion. I usually don't do that, but it may work better for some people. The next section lists the same questions but with short answers. You can use these yourself, or make copies of them to give to the students at the end of the circle as a rough reminder of what was discussed.

It's a good idea to know some of the true distances and sizes so you can give the students a feeling for what a true scale model would be. Here are some numbers (all are averages, and all are given in miles and kilometers):

Diameter of the Earth	7926 mi. (12,756 km.)
Diameter of the Sun	865,000 mi. (1,392,000 km.)
Diameter of the Moon	2160 mi. (3475 km.)
Distance from Earth to Sun	93,020,000 mi. (149,670,000 km.)
Distance from Earth to Moon	238,900 mi. (384,400 km.)

Think of some objects that the students are familiar with that will illustrate these differences.

When you draw figures on the blackboard, for example to illustrate the shapes of the shadows cast by the earth or moon in space, you'll obviously have to exaggerate size differences, et cetera, to make sure that the shadow outlines aren't 50 feet long. But do tell the students about the exaggeration, and draw two or three versions, each with less exaggeration, so they can better visualize the true situation in space.

Of course all the questions below are just suggestions, representing topics of most interest to me. Feel free to add your own. If you have any great suggestions for additional questions that fit the general format here, I'd love to hear about them. Send me email to the address at the top of the first page of this article.

Finally, yes: I know that Pluto is no longer considered a planet. And that with global warming it may soon be impossible to "stand at the north pole".

2 Geography Problems

1. **What is special about the north and south poles?**
2. **At any given time, how much of the earth is sunlit?**
3. **Why is the globe of the earth tilted in any standard mount?**
4. **What does straight up on the globe represent (in a standard mount)?**
5. **What causes seasons? Why are they reversed in the southern hemisphere? Why is it hotter when the sun is higher in the sky?**
6. **What does sunrise look like at the north pole? What does it look like near the north pole?**
7. **What is special about the north star? Why isn't there a south star? If you are sailing in the south pacific and the north star appears on the horizon, where are you?**
8. **If you point a camera at the north star at night and leave the shutter open for a few hours, what does the photo look like?**
9. **What causes the lengths of days to change? What do the arctic and antarctic circles represent?**
10. **What do the tropics of Cancer and Capricorn represent?**
11. **Can the sun ever be directly overhead (here in Northern California)? Where can it be directly overhead? At what time(s) of year?**
12. **How long are the days at the equator? At the poles? In between? How high in the sky is the sun at the north pole on June 21?**
13. **Can the sun ever appear due north?**
14. **A bear walks one mile south, one mile east, and one mile north, and is exactly where he started. What color is the bear? Where is the bear?**
15. **Why are there two hottest times of year in places like Madras, India, midway between the equator and the tropic of Cancer? What do the seasons look like there?**
16. **What do the seasons look like at the equator? (Fact, in some places on earth near the equator, plants do not have a twelve month growing season—some have 10 month seasons, some have 13 month seasons. Why?)**
17. **What is the geometric relationship between the sun and the earth at the solstices, at the equinoxes? What does "equi" mean in "equinox"?**
18. **Where can you take the best sunset photos—in Hawai'i or in Alaska?**
19. **What causes the phases of the moon?**

20. **Why do only planets nearer to the sun than the earth (Venus and Mercury) exhibit phases, and the others (Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto) do not?**

21. **What causes lunar eclipses? What causes solar eclipses?**

22. **Why do you tend to see more lunar than solar eclipses?**

23. **What causes partial lunar or partial solar eclipses?**

24. **Why are there time zones? What is GMT?**

25. **What is the problem with lunar calendars (which ours is, sort of)?**

26. **What is a leap year?**

27. **What is a leap second?**

28. **Estimate the error in days in the 1400s when Pope Gregory fixed the Julian calendar that was started by Julius Caesar in about year 0. Caesar had leap years, but no correction for the fact that the year is really about 365.24 days.**

29. **Look at the Latin roots for September, October, November, December—they are "sept", "oct", "nov", "dec", corresponding to 7, 8, 9, 10. But they are months 9, 10, 11, and 12. Why?**

30. **What would the seasons look like on Uranus (which has an 84 degree axis tilt).**

31. **What would the earth be like with no tilt?**

32. **Why does a compass work? Why doesn't a compass work perfectly?**

3 Geography Problem Hints and Answers

Note: The answers here obviously make certain assumptions that are not exactly true, but they are close enough to true that they explain a great deal. For example, the earth is not perfectly round, there are mountains and things that block your vision of what would be the true horizon on a flat earth, et cetera. If the earth were covered by a calm ocean, you can see that the answers below would be much closer to the real truth (but not perfect—there are tides and waves and things). Also, the orbits of planets and moons are not circles, but are ellipses, some more eccentic than others.

1. **What is special about the north and south poles?**

   The earth spins around an axis, and that axis passes through the north and south poles.

2. **At any given time, how much of the earth is sunlit?**

   Approximately half. When the sun's rays arrive at the earth they are almost parallel and if they were perfectly parallel and the earth were a perfect sphere, exactly half of the surface would be sunlit. Depending on the level of the class, you can consider questions like, "What if the sun were a point source of light?" "The sun is actually quite a bit larger than the earth. How much of the surface is sunlit? What do the shadows look like?" This second question is closely tied to question 21.

3. **Why is the globe of the earth tilted in any standard mount?**

   See the answer to item 4, below.

4. **What does straight up on the globe represent (in a standard mount)?**

   The earth's axis is not parallel to the axis of the solar system. In other words, if you look at the approximate circle that the earth makes as it moves around the sun, the axis of that circle is tilted about 23.5° from the axis of rotation of the earth. As we will see below, this is *extremely* important.

   Also notice that the axis of the earth continues to point in the same direction (relative to the rest of the galaxy) as it moves around the sun. Thus, for example, when the axis coming out of the north pole hits the axis of rotation of the earth around the sun, someone standing on the north pole could see the sun for a entire rotations of the earth (that's the "midnight sun"), and obviously some other guy at the south pole would be unable to see the sun for the same period. When the earth revolves to the other side of the sun, things are reversed, and the guy at the south pole will see the midnight sun.

5. **What causes seasons? Why are they reversed in the southern hemisphere? Why is it hotter when the sun is higher in the sky?**

   As the earth moves around the sun, its axis continues to point in the same direction, so on one side of the sun, for example, the north pole will tilt toward the sun and on the other side, it will be pointed away. Obviously at the north pole there will be a lot more sun when the sun is visible, so that will be the summer there. Near the north pole, the same thing happens, and the closer to the equator you get, the less will be the effect. But any part of the earth north of the equator will see a little more sun than places south of the equator during the summer months in the north.

Clearly, when the south pole is pointed toward the sun (the winter in the north), southern places will get more sunlight, and it will be summer in the south.

Suppose that you live somewhere like northern California, and draw an east-west line through your city that goes all the way around the earth. For any particular position of the earth relative to the sun, a certain amount of this circle will be visible to the sun. If more than half is visible (during the summer months), that means more than half of the 24 hour day will be sunlit. If less is visible (like in the winter), less than half will be sunlit.

It is hotter when the sun is higher in the sky since more radiation hits a unit area of ground. Imagine a uniform pattern of rays coming from the sun and hitting a square of paper. If that paper is tilted, less of its area is exposed to the sun, so fewer of the rays will strike it. The temperature is higher the more rays hit it.

6. **What does sunrise look like at the north pole? What does it look like near the north pole?**

   On approximately the equinox, the sun will appear right on the horizon, and will spin around on the horizon, getting a tiny bit higher each day. At the summer solstice (about June 21), the sun will be looping around as high as it ever gets there—about 23.5° degrees above the horizon. After the solstice, the sun will continue to spin, but will get lower and lower until it disappears below the horizon on about September 22.

   Near the north pole, the sun will peek out for a second and then disappear. The next day, it will peek out for a tiny bit longer. As days go by, it will make little loops, skimming along the horizon, but each day the time above the horizon will be more. Finally, the sun will be completely above the horizon all the way around, but will loop higher in the sky in one direction and lower in the other.

7. **What is special about the north star? Why isn't there a south star? If you are sailing in the south pacific and the north star appears on the horizon, where are you?**

   It is totally due to chance that the north star is aligned with the axis of the spinning earth. It is not perfectly north, but is *very* close to perfectly north. In the southern hemisphere there does not happen to be such a star.

   As you move north from south of the equator, the first time you can see the north star is when you hit the equator. At this point, the north star will be exactly on the horizon.

8. **If you point a camera at the north star at night and leave the shutter open for a few hours, what does the photo look like?**

   You'll get a bright dot in the center where the north star is, and circular star trails caused by all the stars that appear to spin around it.

   A sort of principle of relativity says that things will look the same to you if you are on a spinning earth as they would if the earth were fixed and the stars moved around it. In fact, before Galileo, that's what people thought. So if the earth is fixed and the sky spins around the north star, the photo will look as described in the previous paragraph.

9. **What causes the lengths of days to change? What do the arctic and antarctic circles represent?**

The sun always lights up (roughly) half of the earth. But since the axis of rotation is tilted relative to the sun, at various times of year, the path of your city may have more or less of it in the sunshine. See item 4.

North of the arctic and south of the antarctic circle, there is at least some time when the sun will be invisible for a whole day or more. The closer to the poles you are, the more days of invisibility there will be.

10. **What do the tropics of Cancer and Capricorn represent?**

    Between these two tropics, the sun is directly overhead for at least one day of the year. They are 23.5° north (Cancer) and south (Capricorn) of the equator.

11. **Can the sun ever be directly overhead (here in Northern California)? Where can it be directly overhead? At what time(s) of year?**

    No, the sun can never be directly overhead for a person in California. Only between the tropics of Cancer and Capricorn. See item 10.

    On the equator, the sun will be directly overhead at the equinoxes. At the tropic of Cancer, it will be overhead on about June 21; on the tropic of Capricorn, it will be overhead on about September 22. In between, the sun will be overhead twice; once while it is moving gradually north, and once while it is moving gradually south.

12. **How long are the days at the equator? At the poles? In between? How high in the sky is the sun at the north pole on June 21?**

    On the equator, every day is exactly 12 hours of daylight and 12 hours of night. At the poles, the days and nights are both 6 months long. In between it depends on where you are. If you are in the arctic or antarctic circle, some of the days (and nights) will be longer than a day because the sun is never visible.

    If you are not in the arctic or antarctic circles, the length of the day plus the length of the night is 24 hours, but the relative lengths vary, depending on the time of year. The closer you are to the equator, the closer are the lengths of day and night. If you are near (but outside) the arctic or antarctic circle, the day can be up to 24 hours long and the night as little as zero and vice versa. In between, the lengths are intermediate.

13. **Can the sun ever appear due north?**

    Well sure, from the south pole everything is north. But the answer is yes in general, as long as you live south of the tropic of Cancer.

14. **A bear walks one mile south, one mile east, and one mile north, and is exactly where he started. What color is the bear? Where is the bear?**

    It started at the north pole, so it's a polar bear, and hence white.

    There are actually an infinite number of places on earth where the bear could walk like this—imagine a place near the south pole where the circumference of the earth is 1 mile. Start one mile north of that, and so south a mile, around the earth once (in a mile) and then back north to where you started.

This also works for circles that have circumference $1/2$ mile, $1/3$ mile, $1/4$ mile, and so on (in which case the bear's eastern trek will take him around the earth 2, 3, 4, ..., times). The bear could not be here, however, since there are no bears in Antarctica.

15. **Why are there two hottest times of year in places like Madras, India, midway between the equator and the tropic of Cancer? What do the seasons look like there?**

    See item 11. For locations between the equator and one of the tropics of Capricorn or Cancer, the sun passes directly overhead twice—once on the way north, and once on the way south.

    So in Madras, for example, when the sun heads south and the "winter" begins, days get shorter and shorter until it gets to the maximum point south. Then as it comes north, the days get longer and longer until it is directly overhead. Then it goes even further north, so the days get shorter again, but not for too long, until the sun gets to the northernmost point and heads south again. When it is directly overhead for the second time is the second heat spell of the summer.

    Madras, by the way, is about halfway between the equator and the tropic of Cancer. Locations that are roughly half way up (at about 12° north or south) will feel this effect most strongly.

16. **What do the seasons look like at the equator? (Fact, in some places on earth near the equator, plants do not have a twelve month growing season—some have 10 month seasons, some have 13 month seasons. Why?)**

    At the equator all days are exactly 12 hours of daylight and 12 hours of night. The sun is directly overhead twice, so it is very slightly warmer then.

    But since there is basically no variation in day length, there are no seasons (at least in some places). Obviously, wind or sea currents caused by the seasons elsewhere can cause seasons at the equator.

    Notice that there is a huge biological selective advantage to having a season that is out of sync with the rest of the forest (if you're a plant, that is). There is no way for insects with a different life cycle to specialize on you, since they can't guarantee that the appropriate food will be available every year.

17. **What is the geometric relationship between the sun and the earth at the solstices, at the equinoxes? What does "equi" mean in "equinox"?**

    At the solstices, the axis of the earth is pointed toward the sun as much as possible, either on the north end or the south end. At the equinoxes, it points neither toward nor away from the sun. On an equinox, the day and night are equal at every point on the earth (except right at the poles).

    So "equi" means "equal", of course.

18. **Where can you take the best sunset photos—in Hawai'i or in Alaska?**

    Hawai'i is very close to the equator, so the sun sets almost straight down, so there's a very short sunset, and it is difficult to photograph there. In Alaska, the sun coasts very gently into the horizon, so there are many chances to get a good photo.

19. **What causes the phases of the moon?**

The sun lights only half the moon, since it is a sphere, just like the earth. But we look at the half-lit sphere or the moon from different angles and see various crescents. It's easiest to see this with a model of a sphere and a flashlight in a darkened room.

20. **Why do only planets nearer to the sun than the earth (Venus and Mercury) exhibit phases, and the others (Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto) do not?**

    We can get planets like Mercury and Venus between us and the sun, and hence see phases. But all we can see of the outer planets is the part of them that is facing the sun, so we never see phases.

    Again, it's easiest to visualize this either in your mind's eye, or with a sphere and a flashlight in a darkened room.

21. **What causes lunar eclipses? What causes solar eclipses?**

    A lunar eclipse occurs when the moon gets into the earth's shadow. A solar eclipse is when the earth is in the moon's shadow. The moon is much smaller than the earth, so it's shadow is smaller, so there aren't as many solar eclipses. Also, when a lunar eclipse occurs, anyone who happens to be on the correct side of the planet can see it; for a solar eclipse, you have to be in exactly the right place.

    In fact, the moon's shadow on the earth is usually almost a point, only a few miles wide. The size of the shadow varies, of course, depending on how far away the moon is, and the moon's distance varies because its orbit is not exactly a circle, but rather an ellipse. The earth's shadow is bigger than the moon itself.

22. **Why do you tend to see more lunar than solar eclipses?**

    See item 21.

23. **What causes partial lunar or partial solar eclipses?**

    If the sun is not completely blocked, the eclipse is partial.

24. **Why are there time zones? What is GMT?**

    Becuase the sun is at its highest point at different times all over the earth. To avoid chaos, we generally group together sectors of about 1/24 of the earth into each time zone and define the time to be the same in all those places.

    GMT is "Greenwich Mean Time" – the time at the great observatory in Greenwich, England. If you want to agree on a time with someone who could be anywhere on earth, the easiest thing to do is to agree that no matter where you both are, you'll just use GMT.

25. **What is the problem with lunar calendars (which ours is, sort of)?**

    The moon doesn't go around the earth a nice round number of times in a year—it's roughly 28 days, but not exactly, and even then, 28 does not divide evenly into 365.

26. **What is a leap year?**

    The earth goes around the sun about every 365.24 days—roughly 365 and a quarter. Every four years, you add a leap day to bring things back into synchronization.

27. **What is a leap second?**

    The earth's spinning is always slowing down, and rather than change the length of a second, we add a second every now and then to make things as exact as possible. Leap seconds are getting more and more common (but not enough that any of us would notice).

28. **Estimate the error in days in the 1400s when Pope Gregory fixed the Julian calendar that was started by Julius Caesar in about year 0. Caesar had leap years, but no correction for the fact that the year is really 365.24 days.**

    Well, the error is roughly .01 days per year, so in 1400 years, the error should be about 14 days. Since the real length of a year is slightly less than 365.24 days, the correction was 11 days.

    Today, to avoid this mess, there is a leap year every 4 years, but not on the centuries. But there is one every 4 centuries, but then there isn't one every 1000 years, so with these corrections of corrections of corrections, the time is kept pretty accurate.

29. **Look at the Latin roots for September, October, November, December—they are "sept", "oct", "nov", "dec", corresponding to 7, 8, 9, 10. But they are months 9, 10, 11, and 12. Why?**

    There were only 10 months in the original Roman calendar. Then Julius Caesar added one (and called it July, of course). Then Augustus Ceaser added one and called it August. The funny thing is that since July had 31 days, Augustus couldn't stand the thought of "his" month only having 30 days, so he made August have 31 also. He stole the extra day from poor February.

30. **What would the seasons look like on Uranus (which has an 84 degree axis tilt).**

    Huge arctic and antarctic circles. Lots of the planet experiences a "midnight sun". There are huge seasonal changes.

31. **What would the earth be like with no tilt?**

    There would be no seasons.

32. **Why does a compass work? Why doesn't a compass work perfectly?**

    There's a magnetic field in the earth that is roughly aligned with the spin axis. But it does not point at the north and south poles; it only points near them. So there are compass errors that depend on exactly where on earth you happen to be standing.