Combinatorial Problems

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The problems below are arranged roughly in order of difficulty, so if the first ones are far too easy, jump to the middle, or to the end.

(1) If A, B, and C are cities If there are 4 roads from $(A \Longrightarrow B)$ and 3 from $B \Longrightarrow C$, how many ways are there from $A \Longrightarrow C$? (Assume that all roads are one-way, in the direction of the arrows.)

(2) If, in addition, there are 6 roads from $C \Longrightarrow D$, how many ways from $A \Longrightarrow D$?

(3) As in the problem above, but 4($A \Longrightarrow B$), 3($B \Longrightarrow C$), 5($A \Longrightarrow D$), 5($D \Longrightarrow C$). How many ways from $A \Longrightarrow C$?

(4) How many ways can you choose a captain and co-captain of a football team with 11 members?

(5) How many ways are there to choose a president, a vice-president, and treasurer from a club of 15, assuming all three are different people?

(6) How many ways to put a white and black rook on a chessboard so that neither can attack the other? (Rooks can only attack along rows and columns – not along the diagonals.)

(7) How many ways to put a white and black king on a chessboard so that neither attacks the other? (A king attacks only those squares adjacent to it, so a king away from the edge of the board attacks the 8 adjacent squares.)

(8) If you have an alphabet of 26 letters, how many 3-letter words can you make? What if the three letters all have to be different? How many 5 letter words can you make, if you can repeat letters, but can't have 2 in a row that are the same?

(9) How many rearrangements can be made of the letters in the following words: VECTOR, TRUST, CARAVAN, CLOSENESS, MATHEMATICAL? (For example, for "VECTOR", some possibilities include: VECTRO, OTCEVR, and ROTVEC.)

(10) How many diagonals are there in a convex n-gon?

(11) How many ways can you put 8 mutually non-attacking rooks on a standard 8×8 chessboard?

(12) There are 3 rooms in a dormitory, a single, a double, and a quad. How many ways are there to assign 7 people to the rooms?

(13) How many 10-digit numbers have at least 2 equal digits?

(14) How many ways can you put 2 queens on a chessboard so that they don't attack each other? (Queens attack both on the rows and on the diagonals of a chessboard.)

(15) How many ways can you split 14 people into 7 pairs?

(16) N boys and N girls are in a dance class. How many ways are there to pair them all up?

(17) How many ways are there to choose a team of 3 students from a group of 30?

(18) One student has 6 books and another has 8. In how many ways can they exchange 3 books of the first student for 3 books of the second?

(19) How many ways can a group of 10 boys be divided into two basketball teams of 5 boys each?

(20) Ten points are marked on the plane so that no three of them are in a straight line. How many different triangles can be

formed using these 10 points as vertices?

(21) A group of soldiers contains 3 officers, 6 sergeants, and 30 privates. How many ways can a team be formed consisting of 1 officer, 2 sergeants, and 20 privates?

(22) Ten points are marked on a straight line and 11 on another line, parallel to the first. How many triangles can be formed from these points? How many quadrilaterals?

(23) How many ways can you put 10 white and 10 black checkers on the black squares of a checkerboard?

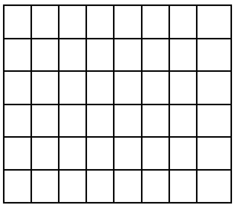
(24) How many 10-digit numbers have the sum of their digits equal to 1? The sum equal to 2? To 3? To 4?

(25) To win the California lottery, you must choose 6 numbers correctly from a set of 51 numbers. How many ways are there to to make your 6 choices?

(26) A person has 10 friends. Over several days he invites some of them to a dinner party in such a way that he never invites exactly the same group of people. How many days can he keep this up, assuming that one of the possibilities is to ask nobody to dinner?

(27) There are 7 steps in a flight of stairs (not counting the top and bottom of the flight). When going down, you can jump over some steps if you like, perhaps even all 7. In how many different ways can you go down the stairs?

(28) The following illustration is a map of a city, and you would like to travel from the lower left to the upper right corner along the roads in the shortest possible distance. In how many ways can you do this?



(29) In how many ways can 12 pennies be put in 5 purses? What if none of the purses can be empty?

(30) In how many ways can you put k identical things into n boxes, where the boxes are numbered $1, \cdots, n$? What if you must put at least one thing in each box (so, of course, $k > n$)?

(31) A bookbinder must bind 12 identical books using red, green, or blue covers. In how many ways can this be done?

(32) A train with M passengers must make N stops. How many ways are there for the passengers to get off the train at the stops? What if we only care about the number of passengers getting off at each stop?

(33) How many ways are there to arrange 5 red, 5 green, and 5 blue balls in a row so that no two blue balls lie next to each other?

(34) How many ways are there to represent 100000 as the product of 3 factors if we consider products that differ in the order of factors to be different?

(35) There are 12 books on a shelf. In how many ways can you choose 5 of them so that no two of the chosen books are next to each other on the shelf?

(36) In how many ways can a necklace be made using 5 identical red beads and 2 identical blue beads?

(37) How many 6-letter “words” contain at least one letter “A” (if any sequence of letters counts as a word)?

(38) Given 6 vertices of a regular hexagon, in how many ways can you draw a path that hits all the vertices exactly once?

(39) Within a table of m rows and n columns a box is marked at the intersection of the p^{th} row and q^{th} column. How many of the rectangles formed by the boxes of the table contain the marked box?

(40) A $10 \times 10 \times 10$ cube is formed of small unit cubes. A grasshopper sits in the center O of one of the corner cubes. At a given moment, it can jump to the center of any of the cubes which has a common face with the cube where it sits, as long as the jump increases the distance between point O and the current position of the grasshopper. How many ways are there for the grasshopper to reach the unit cube at the opposite corner?

(41) Find the number of integers from 0 to 999999 that have no two equal neighboring digits in their decimal representation.

(42) How many ways are there to divide a deck of 52 cards into two halves such that each half contains exactly 2 aces?

(43) How many ways are there to place four black, four white, and four blue balls into six different boxes?

(44) In Lotto, 6 numbers are chosen from the set $\{1, 2, \ldots, 49\}$. In how many ways can this be done such that the chosen subset has at least one pair of neighbors?

(45) Find the sum $S_n = \sum_{k=1}^{n} \binom{n}{k} k^3$. Hint: the sum can be interpreted as the number of ways to choose a committe of at least one that includes a president, vice-president, and treasurer (not necessarily distinct persons) from a set of n people.

(46) Given a set of $3n + 1$ objects, assume that n are indistinguishable, and the other $2n + 1$ are distinct. Show that we can choose n objects from this set in 2^{2n} ways.

(47) In how many ways can you take an odd number of objects from a set of n objects?

(48) n persons sit around a circular table. How many of the $n!$ arrangements are distinct, i.e., do not have the same neighboring relations?

(49) $2n$ points are chosen on a circle. In how many ways can you connect them all in pairs such that none of the segments overlap?

(50) In how many ways can you triangulate a convex n-gon using only the original vertices?

(51) If you have a set of n pairs of parentheses, how many ways can you arrange them "sensibly". For example, if you have 3 pairs, the following 5 arrangements are possible: ((())), (())(), ()(()), (()()), ()()().

(52) Can you arrange the numbers $1, 2, \ldots, 9$ along a circle such that the sum of two neighbors is never divisible by 3, 5, or 7?

(53) In how many ways can you choose two distinct subsets from a collection of N items? The two subsets together need not include all N items.

(54) In how many ways can you select an odd number of objects from a collection of N objects?

(55) How many subsets of the set $\{1, 2, 3, \ldots, N\}$ contain no two successive numbers?

(56) How many ways are there to put seven white and two black billiard balls into nine pockets? Some of the pockets may be empty and the pockets are considered distinguishable.

(57) Consider a circular set of N seats, with a different child sitting in each. How many ways can you rearrange the children so that no child moves more than one seat to the right or the left of his/her original position?

(58) Can you find a polyhedron with an odd number of faces, where each face has an odd number of edges?

(59) Given are 2001 distinct weights, $a_1 < a_2 < \cdots < a_{1000}$ and $b_1 < b_2 < \cdots < b_{1001}$. Find the weight with rank 1001 using at most 11 weighings. (In other words, if all the weights were mixed together and sorted by weight, find the middle one with at most 11 weighings.)

(60) Can you partition the set of positive integers into infinitely many infinite subsets such that each subset is generated from any other by adding the same positive integer to each element of the subset?

(61) Consider all $2^n - 1$ nonempty subsets of the set $\{1, 2, \ldots, n\}$. For every such subset, we find the product of the reciprocals of each of its elements. Find the sum of all these products. (For example, if the set consists of $\{1, 2, 3\}$, the

subsets are $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, and $\{1,2,3\}$. The products of the reciprocals are 1, 1/2, 1/3, 1/2, 1/3, 1/6, and 1/6, respectively. The sum of all of them is 3.

(62) How many ways are there to group 4 pieces of luggage? 5 pieces? (Here are the groupings of 3 pieces, A, B, and C: ABC, $A \mid BC$, $B \mid AC$, $C \mid AB$, $A \mid B \mid C$. The vertical bars represent divisions into groups.)

(63) Find the number of poker hands of each type. For the purposes of this problem, a poker hand consists of 5 cards chosen from a standard pack of 52 (no jokers). Also for the purposes of this problem, the ace can only be a high card. In other words, the card sequence $A\clubsuit$, $2\diamondsuit$, $3\diamondsuit$, $4\spadesuit$, $5\heartsuit$ is not a straight, since the ace is a high card only.

Here are the definitions of the hands:

Royal flush: 10 through A in the same suit.

Example: $10\spadesuit$, $J\spadesuit$, $Q\spadesuit$, $K\spadesuit$, $A\spadesuit$.

Straight flush: 5 cards in sequence in the same suit, but not a Royal flush.

Example: $4\diamondsuit$, $5\diamondsuit$, $6\diamondsuit$, $7\diamondsuit$, $8\diamondsuit$.

Four of a kind: Four cards of the same rank.

Example: $Q\spadesuit$, $Q\heartsuit$, $Q\diamondsuit$, $Q\clubsuit$, $7\spadesuit$.

Full house: Three cards of one rank and two of another.

Example: $3\spadesuit$, $3\diamondsuit$, $3\clubsuit$, $9\heartsuit$, $9\diamondsuit$.

Flush: Five cards in the same suit that are not in sequence.

Example: $3\clubsuit$, $4\clubsuit$, $5\clubsuit$, $6\clubsuit$, $8\clubsuit$.

Straight: Five cards in sequence that are not all in the same suit.

Example: $6\clubsuit$, $7\clubsuit$, $8\diamondsuit$, $9\spadesuit$, $10\clubsuit$.

Three of a kind: Three cards of the same rank; the others of different rank.

Example: $J\clubsuit$, $J\diamondsuit$, $J\spadesuit$, $7\diamondsuit$, $K\heartsuit$.

Two pairs: Two pairs of cards.

Example: $5\heartsuit$, $5\spadesuit$, $8\diamondsuit$, $8\clubsuit$, $A\spadesuit$.

Pair: A single pair of cards.

Example: $3\clubsuit$, $3\diamondsuit$, $5\spadesuit$, $9\diamondsuit$, $Q\diamondsuit$.

Bust: A hand with none of the above.

Example: $2\clubsuit$, $4\spadesuit$, $6\diamondsuit$, $8\heartsuit$, $10\clubsuit$.