Combinations and Permutations

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Permutations: A permutation is an arrangement of things. The number of ordered ways to choose k things from a set of n is $n(n-1)(n-2)\cdots(n-k+1)$. This is because the first object can be chosen in any of n ways, the second in any of $n-1$ ways, and so on.

Example: How many ways are there to choose an ordered set of 3 numbers from a set of 4? **Answer:** $4 \cdot 3 \cdot 2 = 24$. Suppose the numbers are $1, 2, 3$, and 4. The possibilities are listed below. Note that it is very convenient to organize a list like this in some order to be certain you haven't left anything out. In this case, they're arranged in a sort of "alphabetical order", as if $1, 2, 3, 4$ were A, B, C, D in the alphabet:

$$123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432.$$

Combinations: Combinations are like permutations except that they are unordered. In the example above, the combinations of 4 things taken two at a time would not include both 41 and 14. To get the number of combinations of n things taken k at a time, we must divide the number of permutations by $k!$ to get rid of duplicate permutations.

Thus, the number of combinations of n things taken k at a time is:

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

This formula works fine for $n = 0$ and/or $k = 0$, as long as you remember that $0! = 1$.

Example: The number of ways to choose 3 numbers from the set $\{1, 2, 3, 4, 5, 6\}$ is $\binom{6}{3} = 20$. The possibilities include. Notice that this list is also in "alphabetical order":

$$123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456.$$

Relationship with Pascal's Triangle: Here is Pascal's triangle, written both in the usual way, and written with its terms expressed as combinations.

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

In other words, the entries in Pascal's triangle equal to the corresponding entry in the triangle of combination coefficients on the left.

Relationship of Pascal's Triangle with the binomial theorem: Note that in the following expansion, the coefficients of the terms correspond exactly to the fourth row (starting at row zero) of Pascal's Triangle:

$$(a+b)^n = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4.$$

The general expansion of a binomial to a power is given by:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

If you've never done it, a good exercise is to start with, say, the expansion of $(a+b)^3$ and multiply that by hand by $(a+b)$ to get $(a+b)^4$. You'll then see exactly why the rule for calculating entries in Pascal's Triangle corresponds to the combination coefficients.

Counting Tricks: If you have no idea how to proceed, it's often a good idea to count or calculate (very carefully) the first few cases, and then try to work from that. Here's a simple example of that, together with a new technique for analysis. The problem is to find a formula for the sum of the first n squares. In other words, find a formula for $S(n)$, where

$$S(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

In this case, it's easy to work out the first few cases: $S(0) = 0$, $S(1) = 1$, $S(2) = 1 + 4 = 5$, $S(3) = 1 + 4 + 9 = 14$, and so on. Here's a list of the first few (be sure to start your row with the value for $n = 0$):

$$0 \quad 1 \quad 5 \quad 14 \quad 30 \quad 55 \quad 91 \quad 140 \quad 204 \quad 285$$

Now take differences of the numbers in this row, forming a second row. Then take differences in the second row, forming a third, and so on:

$$\begin{array}{ccccccccccccccccccc}
0 & & 1 & & 5 & & 14 & & 30 & & 55 & & 91 & & 140 & & 204 & & 285 \\
& 1 & & 4 & & 9 & & 16 & & 25 & & 36 & & 49 & & 64 & & 81 & \\
& & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & 15 & & 17 & & \\
& & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & & \\
& & & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}$$

If you're lucky and the last row is all zeroes, it's obvious that you can generate the entire top row from the numbers on the diagonal (in this case, $0, 1, 3, 2$). You can write them on the diagonal, extend the final 2 forever, and fill in any other term by adding the term to its left to the term below and to the left.

But there's also a formula. In this case, it is:

$$0\binom{n}{0} + 1\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}$$

$$= 0 + 1n + \frac{3n(n-1)}{2} + \frac{2n(n-1)(n-2)}{6} = \frac{n(n+1)(2n+1)}{6}.$$

Counting Mountains — Catalan Numbers: Using n pairs of the characters / and \, build as many chains of mountains as you can. For $n = 1, 2, 3,$ and 4 here are the possibilities:

```
n=1 /\
```

```
     /\
n=2 /  \  /\/\
```

```
      /\
     /  \    /\        /\    /\/\
n=3 /    \  /  \/\  /\/  \  /    \  /\/\/\
```

```
       /\
      /  \      /\/\      /\        /\          /\
     /    \    /    \    /  \/\    /  \      /\/  \    /\/\/\    /\/\
n=4 /      \  /      \  /      \  /    \/\  /      \  /      \  /    \/\

                            /\
     /\  /\     /\           /  \      /\/\      /\          /\
    /  \/  \  /  \/\/\  /\/    \  /\/    \  /\/  \/\  /\/\/  \  /\/\/\/\
```

So there are 1, 2, 5, and 14 ways to construct legal mountain chains for 1, 2, 3, and 4 pairs of the slash characters. How do we count these?

The answer is

$$\frac{1}{2n+1}\binom{2n+1}{n}.$$

The way to see this is to imagine all possible arrangements of $n+1$ up-strokes and only n down-strokes. From among these $n+n+1 = 2n+1$ strokes, we need to choose n of them to be up-strokes, so there are $\binom{2n+1}{n}$ choices. But imagine putting these stroke sequences next to each other. For $n = 3$, look at the example “up down down up up up down”:

```
                                 /\
                          /\/\  /
                   /\/\  /    \/
            /\/\  /    \/
     /\/\  /    \/
/\  /    \/
  \/
```

All of these will form a sort of staircase as above, and all will have a lowest point marking the lowest edge. Since the low point can occur at any of the $n+n+1$ positions, we must divide by $2n+1$.

These are known as the Catalan numbers, and they come up in many counting problems. Is it obvious why the number of valid arrangements of n pairs or parentheses is also given by the Catalan numbers? For $n = 3$:

```
((()))  (())()  (()())  ()(())  ()()()
```