Introduction to the Mathematics of Zome

Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
July 1, 2008

Abstract

This article provides a gentle introduction to the Mathematics of Zome, using a set of graded exercises. To use this material, each student, or group of students, will need a small set of Zome parts, including all three lengths of struts that are red, blue and yellow.

Zome parts are available online from `www.zometool.com`.

How To Use This Article

The best way to use it, if you are studying by yourself, is to struggle with the exercises and a set of Zome parts and then check the answer in the second part of the article. If you are teaching a class of younger students using this as a worksheet, you can duplicate only the first part so the students can't just look up the answers. Even if you're happy with your solution, read the official one, since there might be insights there that you missed. Similarly, as your students solve the problems, make sure they've learned all they can at each stage.

To understand the elementary mathematics covered here, you need to know a little geometry (similar figures, the Pythagorean theorem), and a little algebra, up to and including how to solve a quadratic equation and how to manipulate relatively simple algebraic equations.

Zome is seductive, and it's easy for kids (and adults) to ignore the math and just to try to "build stuff". Try to get the students (or yourself, for that matter, if you're trying to learn some math) to work through most of the exercises. But they (and you) will probably enjoy the process a bit more if they're allowed to unleash some of their creativity in Zome construction.

There are hundreds of other ways to use Zome in the classroom, and if you go to the Zome website `www.zometool.com` you can find a great deal of material suitable for classroom use. In this worksheet we work almost entirely with two-dimensional structures built from Zome, but one of the main beauties of Zome is how well it works as a three-dimensional construction set.

Part I

Questions

In what follows, you will use 10 different Zome parts: the little balls into which the struts can be pushed, and three lengths each of each colored strut, red, blue and yellow. We will refer to the blue struts as b_1, b_2 and b_3, where b_1 is the shortest and b_3 is the longest. In a similar way, we will refer to the red struts as r_1, r_2 and r_3 and the yellow struts at y_1, y_2 and y_3. Always, the lengths increase with increasing subscripts.

We will be a little sloppy here and use the names $b_1, b_2, \ldots, y_3$ to refer not only to the struts themselves, but to the lengths of the struts.

When you build a "mathematical" Zome structure, the struts must be straight. It is certainly possible, especially in structures with longer struts, to bend the struts a little. All the mathematics we will explore here depends on the struts being straight.

The main purpose of the exercises contained here is to show how Zome parts can be used to demonstrate mathematical facts. We will construct "proofs" that are perhaps not rigorous mathematical proofs, but can demonstrate certain facts in a very convincing way. In this article, we will call these "proofs by Zome".

In the different problems, we will investigate such things as the relative lengths of the Zome struts, the angles they allow, and the kinds of structures, both two-dimensional and three-dimensional that are possible. If you have a lot of free time, you can probably investigate those relationships without reading any farther. The material below will lead you more rapidly to the results.

1 Preliminary Exercises

1. Examine a Zome ball and the various struts. Note that blue struts always attach to rectangular holes, yellow to triangular holes, and red to pentagonal holes. Also note that every triangular hole on a ball is "equivalent" to every other in the sense that if another is rotated into its place, all the other holes will match up. It's perhaps easier to see this with two Zome balls. The same thing can be said for the other holes: check this to make sure you see why it is true.

2. Build an equilateral triangle using 3 of the b_1 struts. Is it possible to do the same thing using three of the y_1 or three of the r_1 struts? What does this tell you about the angles between pairs of triangular Zome holes or pentagonal Zome holes?

3. What other regular polygons can you form using only blue struts, all having the same length? Squares? Pentagons? ... Can you form a regular polygon from red struts or yellow struts? How can you verify that you have all the answers?

4. To be precise, it is not the exact lengths of the struts that concern us. If the struts and balls were mathematically perfect, the length of, say, strut b_1 would be the distance between the centers of two Zome balls attached tightly to the ends of the strut b_1. Do you see why this is the most sensible definition of length?

5. Can you "prove" that $b_1 + b_2 = b_3$? Does the same relationship hold for the red and yellow struts?

6. Build a triangle using b_1, b_1 and b_2. Next build another triangle with b_2, b_2 and b_3. What do you notice about the two triangles? Can you derive, from that observation:

$$\frac{b_1}{b_2} = \frac{b_2}{b_3}.$$

7. From this point on, we will call the length of the b_1 strut 1, or in other words, $b_1 = 1$ Using the formulas for blue struts derived in the previous two exercises, can you calculate the length of the b_2 strut? Can you calculate the length of the b_3 strut? We will give the name τ (the Greek letter "tau") to the value of b_2.

8. Show that $\tau^2 = 1 + \tau$. This provides a simpler form for the value of b_3. What is it?

9. Since the red and yellow struts satisfy formulas similar to that of the blues; namely, that $r_1 + r_2 = r_3$ and $y_1 + y_2 = y_3$, we suspect that the ratios of y_2/y_1, y_3/y_2, r_2/r_1 and r_3/r_2 are all also equal to τ. Can you construct "proofs by Zome" of these facts? **Hint:** Build triangles b_1, r_1, r_1 and b_2, r_2, r_2 and use similarity. Can you do the same thing for the yellow struts?

10. An interesting class project would be to make a catalog of all possible triangles that can be formed using three struts and three Zome balls. Since the ratios $r_1 : r_2 : r_3$, $b_1 : b_2 : b_3$ and $y_1 : y_2 : y_3$ are all identical, consider only triangles where at least one of the struts is an r_1, a b_1 or a y_1. It might be a good idea to have one master list on the blackboard that different students or groups of students can add to. Make sure that you have some way to detect duplicates. For example, the triangles b_1, r_1, r_1 and r_1, b_1, r_1 are equivalent. How could you be certain that your list contains all of them?

2 The Arithmetic of τ

We noticed already that the value of τ, the golden ratio, satisfies the following equation:

$$\tau^2 = 1 + \tau.$$

1. Can you find a simple formula for τ^3 that involves only constants and τ, but no τ^2 terms?

2. Using the same technique, find a similarly simple formula for τ^4, for τ^5, for τ^6.

3. Find a general formula for τ^n, where $n >= 0$. **Hint:** Remember the Fibonacci numbers F_n: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. They are defined by the equations: $F_0 = 0$, $F_1 = 1$ and if $n > 1$, $F_n = F_{n-1} + F_{n-2}$.

4. (**Extra credit, and a bit more difficult**) See if you can work out the values of τ^{-1}, τ^{-2}, τ^{-3}, et cetera, using the same technique.

3 Relative Strut Lengths

From the previous section, we know the relative strut lengths of the red, blue and yellow struts, at least compared to others of the same color, and that is:

$$r_1 : r_2 : r_3 = b_1 : b_2 : b_3 = y_1 : y_2 : y_3 = 1 : \tau : \tau^2,$$

but now we would like to find out how long the red and yellow struts are in comparison to the blues.

1. Build a triangle using a b_1, a b_3, and two y_2 struts plus four Zome balls. Obviously, one side will need to be made from two struts. Use this triangle to determine the length of the y_2 strut in comparison to the b_1 strut which we have agreed has length one.

2. Using the same idea as above, construct a triangle using a b_1, a b_2, and two r_1 struts plus four balls. Apply similar calculations to find the lengths of the r_1, r_2 and r_3 struts relative to a b_1 strut whose length is 1. **Note:** The result will not be quite as pretty as for the yellow strut lengths.

4 Zome Angles

In this section we will try to determine the angles formed by pairs of struts coming out from a Zome ball. For students who know the law of cosines, it is easy to combine the information we have about the strut lengths and the structure of some of the triangles we formed in Section 1 to determine many angles directly.

The law of cosines for a triangle with sides a, b and c and having angles A, B amd C opposite those angles, respectively, states that:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

If we use as an example the isosceles triangle formed from the three struts b_1, b_2, b_2, if we want to find the measure of the vertex angle, the law of cosines tells us that:

$$1^2 = \tau^2 + \tau^2 - 2\tau^2 \cos C,$$

which we can solve to obtain:

$$C = \arccos\left(\frac{2\tau^2 - 1}{2\tau^2}\right) = \arccos\left(1 - \frac{1}{2\tau^2}\right).$$

Since we can show that $1/\tau^2 = 2 - \tau$ (see the last problem in Section 2) we can see that:

$$C = \arccos\left(1 - \frac{1}{2}(2 - \tau)\right) = \arccos(\tau/2) = 36^\circ.$$

But it would be nicer not to have to use such sophisticated mathematics.

1. Find a simple Zome structure that “proves” that the vertex angle in the isosceles triangle b_1, b_2, b_2 is 36°.
2. If you stick two blue struts into a Zome ball, what are all the possible angles they could make with each other? **Hint:** There are seven of them.

Part II

Solutions

1 Solutions: Preliminary Exercises

1. No solution required.

2. It is impossible to build an equilateral triangle out of three red or yellow struts all having the same length. The Zome balls do not admit having two red struts or two yellow struts attached to a ball at a 60° angle. There are blue holes that form this angle.

3. With a set of equal-length blue struts, you can make an equilateral triangle, square, and regular pentagons, hexagons and decagons (a decagon is a 10-sided figure). No other regular polygon is possible. It is possible to make figures that look almost regular from other colored struts. For example, you can make a loop of 10 yellow struts of the same length that is almost a regular decagon, but the Zome balls in the resulting figure do not lie in a plane: they alternate up and down. Similarly, an almost-flat hexagon can be made of 6 equal red struts.

4. The actual Zome struts are not mathematical lines and the Zome balls are not mathematical points which we would ideally like them to represent. If we imagine that the struts represent ideal lines that pass through their centers, then the point representing the intersection of two of these ideal lines in a Zome ball would lie in the center of that Zome ball.

5. The proof consists of sticking a b_1 and b_2 strut into the same ball so that they form a straight line segment. Compare the length of that combined segment with the length of the longer b_3. Exactly the same thing can be done with the yellow and red struts. (To be technically correct, since we have defined the length of a strut to be the distance between the centers of two balls on the ends of the struts, we should attach balls to the ends of the b_3 strut as well as to the ends of the $b_1 + b_2$ strut combination and compare the lengths of those.)

6. Since the two triangles you built with the blue struts have the struts coming out of equivalent holes in the Zome balls, all the angles in the two triangles must be equal. By AA similarity, the two triangles are similar, and the formula is simply a consequence of the similarity of the triangles.

7. Since $b_1 + b_2 = b_3$ we can substitute $b_1 + b_2$ for b_3 in the formula:

$$\frac{b_1}{b_2} = \frac{b_2}{b_3}$$

to obtain:

$$\frac{b_1}{b_2} = \frac{b_2}{b_1 + b_2}.$$

Since we've agreed that the length of b_1 is 1, we have:

$$\frac{1}{b_2} = \frac{b_2}{1 + b_2},$$

so $b_2^2 = 1 + b_2$. Solving this quadratic equation yields two roots:

$$b_2 = \frac{(1+\sqrt{5})}{2} \quad \text{or} \quad b_2 = \frac{(1-\sqrt{5})}{2}$$

The second root is negative, so since b_2 is obviously possible, it must have the value on the left above, which we will call τ

Numerically, τ is approximately 1.618033987 and it is also known as the "golden ratio" whose value is exactly:

$$\frac{1+\sqrt{5}}{2}.$$

To obtain the value of b_3, just remember that $b_3 = b_1 + b_2$, so $b_3 = 1 + \tau$.

8. Since τ is the root of the quadratic equation $x^2 = 1 + x$, it must be true that $\tau^2 = 1 + \tau$. This means that the length of $b_3 = 1 + \tau = \tau^2$.

9. Since the triangles formed from b_1, r_1, r_1 and b_2, r_2, r_2 are obviously similar, we can just write down that $b_2/b_1 = r_2/r_1$. This means that the ratios of the lengths of the shorter red struts is the same as the ratio of the lengths of the two shorter blue struts, namely: τ. A similar triangle can be built with b_3, r_3, r_3, and by similarity, the ratios of the lengths of the red struts are the same. Exactly the same Zome proof works for the yellows, but use the triangles: b_1, y_1, y_1, b_2, y_2, y_2 and b_3, y_3, y_3.

10. Here is a list of all possible Zome triangles constructed from three struts and three balls that include at least one side of length r_1, b_1 or y_1:

b_1, b_1, b_1	b_1, b_1, b_2
b_1, r_1, r_1	b_1, y_1, y_1
b_1, y_2, y_2	b_1, b_2, b_2
b_1, y_1, r_2	y_1, y_1, b_2
y_1, r_1, y_2	r_1, r_1, b_2
r_1, r_2, y_2	r_1, b_2, y_3
r_1, b_3, y_3	y_1, b_2, r_2

A few of the triangles above are illustrated in Figure 1.

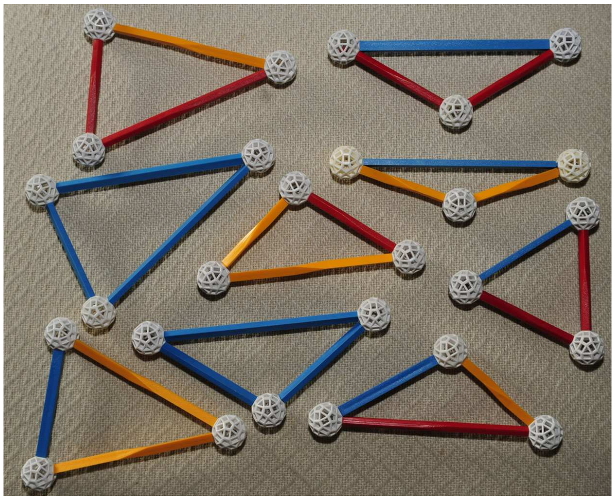

Figure 1: Some Zome Triangles

2 Solutions: The Arithmetic of τ

1. Since $\tau^2 = 1 + \tau$, we can multiply both sides by τ to obtain:

$$\tau^3 = \tau + \tau^2.$$

But we can re-use the original equation and substitute $1+\tau$ for τ^2 in the equation above to obtain:

$$\tau^3 = \tau + 1 + \tau = 1 + 2\tau.$$

2. Since we just worked out the value of τ^3 above, we can multiply both sides by τ to obtain a formula for τ^4:

$$\tau^4 = \tau + 2\tau^2.$$

Again, we'd like to get rid of the τ^2 term, so as before, substitute $1+\tau$ for τ^2:

$$\tau^4 = \tau + 2(1+\tau) = 2 + 3\tau.$$

We can use the same method to obtain:

$$\tau^5 = 2\tau + 3\tau^2 = 2\tau + 3(1+\tau) = 3 + 5\tau,$$

and

$$\tau^6 = 3\tau + 5\tau^2 = 3\tau + 5(1+\tau) = 5 + 8\tau.$$

3. If you don't yet see the pattern, work out a few more and you can make a table (where we have included the coefficients of the constant term and of τ, even if they are 0 or 1):

$$\begin{aligned}
\tau^1 &= 0 + 1\tau \\
\tau^2 &= 1 + 1\tau \\
\tau^3 &= 1 + 2\tau \\
\tau^4 &= 2 + 3\tau \\
\tau^5 &= 3 + 5\tau \\
\tau^6 &= 5 + 8\tau
\end{aligned}$$

Notice that the coefficients are all Fibonacci numbers, and it appears that the general form is this:

$$\tau^n = F_{n-1} + F_n\tau.$$

This can be proved by induction. If $n = 1$ we have $\tau^1 = F_0 + F_1\tau = 0 + \tau$, which is correct. Now assume it is true for $n = k$:

$$\begin{aligned}
\tau^k &= F_{k-1} + F_k\tau \\
\tau^k \cdot \tau &= F_{k-1}\tau + F_k\tau^2 \\
\tau^{k+1} &= F_{k-1}\tau + F_k(1+\tau) \\
\tau^{k+1} &= F_k + (F_{k-1} + F_k)\tau \\
\tau^{k+1} &= F_k + F_{k+1}\tau
\end{aligned}$$

The final line is in the correct form for the $(k+1)^{\text{st}}$ term, so we are done. Note that, as we did as we were experimenting, we substituted $1+\tau$ for τ^2, and to obtain the last line from the next-to-last, we used the definition of the Fibonacci numbers.

4. To obtain the value of $\tau^{-1} = 1/\tau$ is simple. Just divide both sides of the equation $\tau^2 = 1 + \tau$ by τ and rearrange to obtain:

$$\tau^{-1} = \frac{1}{\tau} = -1 + \tau.$$

Divide both sides of that by τ to obtain:

$$\tau^{-2} = -\frac{1}{\tau} + 1,$$

but we know that $1/\tau = -1 + \tau$ so we can substitute that to obtain:

$$\tau^{-2} = 2 - \tau.$$

We can continue to obtain a few more values.

Next, consider the Fibonacci numbers that start at 0. Could you extend them to make a "reasonable" formula for F_{-1}, F_{-2}, F_{-3}, et cetera? For example, we would want $F_{-1} + F_0 = F_1$, and since $F_0 = 0$ and $F_1 = 1$, then F_{-1} would have to be equal to 1. This would force F_{-2} to be equal to -1. Do you see why? With these extensions, would our previous formula work when extended to negative exponents of τ and to Fibonacci numbers with negative indices? In fact, it would, and it is just an exercise in algebra to show that it is.

3 Solutions: Relative Strut Lengths

1. The triangle built using a b_1, a b_3 two y_2 struts is a right triangle whose legs have lengths 1 and τ^2, and whose hypotenuse (since it's made of two y_2 struts) is $2y_2$.

   The Pythagorean theorem tells us that:

   $$(2y_2)^2 = 1^2 + (\tau^2)^2 = 1 + \tau^4.$$

   From the previous section, we know that $\tau^4 = 2 + 3\tau$, so:

   $$4y_2^2 = 3 + 3\tau = 3(1 + \tau) = 3\tau^2.$$

   Dividing by 4 and taking square roots of both sides, we determine that:

   $$y_2 = \frac{\sqrt{3}}{2}\tau,$$

   and since we know that $y_1 : y_2 : y_3 = 1 : \tau : \tau^2$ we can conclude that:

   $$y_1 = \frac{\sqrt{3}}{2}, \qquad y_2 = \frac{\sqrt{3}}{2}\tau, \qquad y_3 = \frac{\sqrt{3}}{2}\tau^2.$$

   If you know the Pythagorean theorem in three dimensions, then there is an easier way to calculate the length of the yellow struts. Construct a cube using eight Zome balls and twelve b_1 struts. You will find that the long (space) diagonal of the cube is exactly the same as the length of two y_1 struts. The three-dimensional version of the Pythagorean theorem tells us that:

   $$(2y_1)^2 = b_1^2 + b_1^2 + b_1^2 = 3b_1^2 = 3,$$

   from which it is easy to calculate that $y_1 = \sqrt{3}/2$.

   You can obtain the three-dimensional version of the Pythagorean theorem by imagining a line connecting a green strut across the diagonal of a face of the cube such that one end of it shares a ball with one end of the double-length y_1 strut. (Note: if you have a set of the green struts, you can do this physically, but many Zome sets do not contain any green struts.) Then the usual version of the Pythagorean theorem can be applied twice: once to show that the length of the green strut is $\sqrt{2}$ and then using a right triangle that includes the green, double-yellow, and a blue strut to show that the double-yellow strut has total length $\sqrt{3}$.

2. As above, we have a right triangle whose sides have lengths 1 and τ and whose hypotenuse is $2r_1$. The pythagorean theorem gives us:

   $$(2r_1)^2 = 1^2 + \tau^2 = 1 + 1 + \tau = 2 + \tau.$$

Take the square root of both sides and divide by 2 to obtain:

$$r_1 = \frac{\sqrt{2+\tau}}{2},$$

and the fact that $r_1 : r_2 : r_3 = 1 : \tau : \tau^2$ allows us to conclude that:

$$r_1 = \frac{\sqrt{2+\tau}}{2}, \qquad r_2 = \frac{\sqrt{2+\tau}}{2}\tau, \qquad r_3 = \frac{\sqrt{2+\tau}}{2}\tau^2.$$

4 Solutions: Zome Angles

1. The simplest is probably the star-shaped structure formed on the left in Figure 2. The ten Zome struts coming out of the central ball obviously make the same angle with each other and they are all co-planar, so they must divide the $360°$ into 10 equal parts, so the angle between any adjacent pair is $36°$. It's easy to check that the angle formed in the triangle uses struts coming out of the same pairs of holes. This also shows, by the way, that the base angles are $72°$, since they are the same as the the angles formed by struts that are two apart on the left in Figure 2.

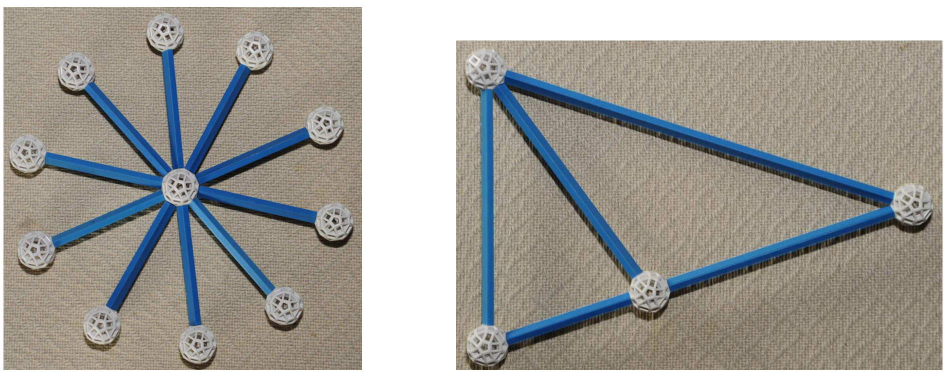

Figure 2: Blue Strut Angles

   There are lots of other possibilities, one of which is illustrated on the right in Figure 2. In this figure, there are two obviously similar triangles and since the b_2, b_2, b_3 sub-triangle is isosceles, we can easily see that the base angle of the largest triangle is bisected by the apex angle of the smallest isosceles triangle. Since the three angles of any triangle must add to $180°$, if the smallest angle is x, we have: $x + 2x + 2x = 180°$, or $x = 36°$.

2. $36°$, $60°$, $72°$, $90°$, $120°$, $154°$, and $180°$.