A Better Mathematics Curriculum

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1 My Motivation

All these ideas have been churning around in my head for years. I've finally written down this draft containing some of them. Although it is very rough, and I'm sure I've left out many obvious things, I would appreciate comments.

The curriculum I'm proposing is aimed at high-school and below, although obviously some of the ideas make sense in a university setting.

Just to show that I'm not completely ignorant of the situation, here is my background: I have a BS in mathematics from Caltech, and a PhD in mathematics from Stanford. I have taught courses in mathematics and computer science at various universities ranging from Stanford to junior colleges. I have done a great deal of volunteer tutoring of mathematics, both for kids and adults who have a great deal of difficulty to kids who are far smarter than I am and have competed on various United States Olympiad math teams. I also did a post-doc at Stanford in electrical engineering and have worked in industry as a software engineer for 20 years.

2 The Problem Today

I think the problem with mathematics education at the university level in this country is that it's generally taught by and aimed at mathematicians. This trickles down to the primary and secondary schools since the committees that determine the curricula are usually packed with university-level mathematicians.

We are doing a great disservice to most of the population who say, quite rightly, "Why did I waste my time learning algebra and trigonometry? I've never solved a single quadratic equation or used the law of sines since I got out of school."

I was trained as a professional mathematician, and in my non-mathematician, non-engineering life, I have never really needed to solve a quadratic equation either.

You would think that in careers that use mathematics heavily–for chemists, physicists, engineers, and computer scientists–at least they are learning the right stuff, but I don't think that's the case. When I took third-year physics as a junior I had to work with Fourier series using actual sines and cosines. I had, of course, learned a great deal about Fourier series in my math courses, but all at a highly theoretical level. We didn't use sines and cosines; we used "complete sets of orthonormal functions on a measurable space" or something. We learned about the weird convergence properties of the functions that were on the edge of not having a Fourier expansion. The bottom line was that I had to teach myself to do it in the "usual case" with sines and cosines applied to reasonably well-behaved functions.

Later in life, I've run into dozens of similar cases, where I had learned the abstract, theoretical theorems, but had never tried to apply them to real problems. If you read *Concrete Mathematics* by Knuth, Graham, and Patashnik, Knuth states in the preface that the reason the book and the course based on it were written and taught is that there were large areas of mathematics he had never seen taught and that he wished he had known in order to do his work in computer science.

In many universities, in fact, engineering departments offer their own math courses since their students are unable to solve engineering problems with the tools they learn from the math department. In the case of the University of Rochester a few years ago, the administration decided basically to eliminate the math department and replace it with service courses for students in various other areas. It didn't happen, but it sure came close to happening. I believe the main impetus was complaints from the engineering departments.

3 What We Should Be Teaching

The bottom line is that we need to teach students the mathematical techniques they will need to use later in life to solve the sorts of problems they will encounter. This obviously varies from person to person, so in the best of all possible worlds, we could teach a different sort of math class to future mathematicians, to future engineers, future computer scientists, carpenters, accountants, or housewives, giving each exactly what they need.

Having a separate curriculum for every type of person is clearly an impossible goal, but I think we can do far better at designing a curriculum that would work well for everyone. In the first few years, it would cover what everyone, technical or non-technical, needs to know. At that point, most folks could stop taking math, and courses more aimed at technical people but not necessarily mathematicians could be taught. I have found that if I have a good idea of how to solve practical problems in an area, it's not hard to abstract it to "pure mathematics", so by the time a person has decided that he/she is going to be a professional mathematician, courses in pure mathematics could be taught.

The argument is made that it's good for the students to learn to think logically, and that doing abstract math is a good way to teach logical thinking. The ability to think logically is critical, but there are plenty of places where you can apply logic to practical problems that require a more concrete type of mathematics.

I fully agree that there are some wonderful, beautiful theorems and results in pure mathematics, some practical and some less so. I have no objection to teaching optional courses on this material for non mathematicians, sort of like I might take an art appreciation course even though I can't draw a recognizable stick figure of a human by myself.

So what should the courses contain (in my humble opinion, of course)?

3.1 Basic Math

Since the bottom line of an education is to allow the student to be able to function well in society, the best approach is to list the sorts of problems with some mathematical content that ordinary people face. With such a list in mind, it's much easier to see what mathematics is useful and what is not. I make no claim that the list below is in any way complete, but I believe that it gives a general idea of the things I have in mind.

- **Household Finances** Calculating change, adding assets, making and keeping a budget, calculating interest (saving or borrowing), calculating taxes and tips, working out your salary from pay rates. Is a car lease better than a car loan? If you borrow $200,000 to purchase a house at 6% interest, about how much will your monthly payment be?

- **Problem-Solving Skills** This is a sort of a universal skill, but I'm not positive how to teach it, and at what level it should be taught to the average non-technical person. Surely the ideas of getting an estimate of the total cost of a project before beginning, or of knowing how to gather data to do research on the cost/benefit of purchasing a house, boat, car, et cetera, is important.

- **A Bullshit Detector** Politicians, salesmen, and con-men may all try to fool you with faulty statistics, bad logic, emotional appeals, et cetera. How do you recognize these? Basically, this amounts to how to apply logic to everyday life. Almost everyone knows what a logical argument *sounds* like–it contains words like "if", and "therefore" in it–but very few people can make one or recognize one. Logical fallacies should be studied.

  Certainly a good topic in this area is the ability to use the resources of the internet while having some sense as to the quality of the information obtained. Blindly accepting the information on web pages may yield higher quality results than fishing in a toilet, but only just slightly.

- **Basic Numeracy** "Numeracy" in mathematics means the same thing as "literacy" does in reading. How can you think about large numbers and small numbers? If a new fighter jet costs a billion dollars, how much is that really, and what sorts and quantities of other things could be purchased for the same amount?

- **Estimation** Be able to estimate roughly what the sum, difference, product or quotient of numbers should be. Estimating costs of loans, amounts of fertilizer needed to cover the yard, amount of paint needed for the walls. Estimating what you'll have to pay for your full grocery cart, how much you'll pay for gasoline, heating oil, water, electricity per month. Estimating how long it will take to drive from Denver to San Francisco.

- **Estimating Probability and Statistics** Being able to understand the relative risks of certain situations. Is it worth it to purchase health insurance, life insurance, different kinds of car insurance? How dangerous is it to ride a motorcycle, to bungee-jump, to smoke cigarettes? How dangerous is it, really, to take a one-in-a-million chance of getting killed? How dangerous is it to do so every day? Should I bet on the lottery?

- **Visualization** How to interpret maps, charts, graphs, tables, et cetera. How to display your own numerical information in a way that is easy for others to understand.

- **Miscellaneous** Doubling recipes. Purchasing beer for a group of 100. Exponential growth: investments, loans, population growth, the national debt, et cetera. How to read and interpret graphs and charts.

The basic course I envision would cover the mathematics and logic necessary to solve (or approximately solve) all the problems above. I have no objection to using calculators heavily, but I would insist that each student be able to estimate, with reasonable accuracy, the results of such calculations before doing them.

It may seem like a small set of topics, but I think they should be taught over and over, with more difficult problems each time around. That way, kids who didn't "get" fractions the first time around would not be doomed to be lost forever–fractions would come up again.

I do not see a need to have courses that teach kids to use computer programs such as email, word processors, or internet browsers–these are easy to use, and besides, they change every year.

3.2 Technical Mathematics

It's a little tougher to make a good list of typical problems for this area, since various careers require different sorts of mathematics. An engineer, surveyor, accountant and computer scientist use very different mathematical tools. I believe that the current curriculum could be improved, even for technical people.

Here is a different sort of list from the one in the previous section:

- **Algorithms** This is basically problem solving. What data do I need to collect? What do I do with it when I have it? How efficient is the calculation? Students should have a good idea of what computer programs are capable of doing, and probably the only way to learn this is to write programs to implement algorithms in some language. I don't think the particular language is too important. In five years it will have changed anyway–consider what's been the "best" language for schools so far: BASIC, PASCAL, C, C++, JAVA–tomorrow it will be something else.

- **“Word Problems”** This refers to the process of taking a problem expressed in natural language and converting it to an algorithm. The problems should, if possible, be more practical. You’ll never get most people to care “how old Mary’s mother will be when she is 5 times as old as Mary is now”.

  I think problems can be more realistic in other ways as well–some should contain extraneous information, some should not contain enough information (in which case the correct answer is “there’s not enough information to solve this problem”). Some should simply ask what information is necessary. All should require not only a numerical answer but a description of why that particular method of calculation is used.

  There should be a strong emphasis on checking the answers, either by plugging them back into the problem to see if they make sense, or finding alternative ways to solve them. In addition, the best problems would require the kids to make an estimate of the solution before even beginning work to serve as a sort of sanity check on the final result.

- **Algebra, Trigonometry, Geometry** Every technical person needs to know how to manipulate equations–to work with unknowns as if they were numbers, to perform the basic algebraic operations to separate variables, how to draw and interpret graphs of equations, et cetera.

  The same goes for trigonometry and geometry. Technical people need to know how to work with angles and to calculate lengths from them, to measure areas and volumes–not just of simple figures like rectangles, triangles, and circles, but of oddball shapes that can be subdivided into more basic ones or can be somehow approximated.

- **Numerical Approximations** Most equations from the real world do not have exact solutions, but it’s important to have some idea how to get approximate solutions and to know roughly the size of the error that’s involved. Rather than spend an eternity on the shapes and problems that can be solved exactly, look at many problems that can’t. (Algebra books are crammed with quadratic equations and 3-4-5 triangles, but cubic, quartic, and quintic equations are ignored, as well as 4-5-6 triangles.)

  I don’t think it’s worthwhile to learn all the different techniques, but it’s important to know how to look up those techniques for your particular problem.

- **Calculus** This is the sort of thing engineers, physicists, and chemists will need. Computer scientists will probably not use it directly much, but if they help the engineers solve their problems, they’d better have a good understanding of the sorts of problems the engineers face. The current calculus syllabus is, as in the other subjects, aimed primarily at the problems that can be solved exactly–at those integrals that can be evaluated. In the real world, very few of them can, just as very few differential equations have an analytic solution, but it’s important to know the properties of the solution and how to approximate that solution as accurately as desired. A heavy emphasis on proof is not required.

- **Concrete Mathematics** Future computer scientits need even more work in the algorithm area. They need to know how to estimate the efficiency of an algorithm, to know whether the algorithm works in all cases, et cetera. The concepts of data storage, error checking, and the problems with using finite-precision numbers need to be emphasized. This information is quite valuable for the engineers et. al., since they will no doubt be using computers to solve their numerical problems.

I think this is close to what is taught today in high schools, although perhaps a few more practical problems would be in order. Students should learn algebra, especially the idea of translating "word problems" into equations. I think perhaps less effort needs to be expended on learning the techniques for solving equations, but general principles can be pounded in, like, for example, the idea that if there are 5 unknowns it's likely that you'll need 5 equations.

In the university, there need to be more practical courses, perhaps aimed at specific careers. Mechanical engineers clearly need to know how to do different mathematics than the physicists or the accountants, or the computer scientists. On the other hand, if I know I'm going to be a mechanical engineer, I can learn just that subset of math. Although I need to know how to solve certain differential equations, I really don't want to go to a pure math theoretical discussion of differential equations in general. I want to practice with particular ones. I also want to know how to use a computer to get numerical answers since most real problems will not be solvable by the analytic solutions that the mathematicians teach.

3.3 Pure Math

I don't think a future pure mathematician will suffer at all if she takes the mathematics for technical people outlined in the previous section. It is important to have a solid understanding of the way mathematics is used in the real world if you're going to do research in pure math. The only thing that's missing is perhaps a very solid idea of what it means to do a mathematical proof.

The concept of proof is currently introduced in geometry courses, and I think in some ways that's the worst place to do it. It's done that way, of course, because that's how we've always done it. There are three problems with this:

1. Geometry is a new subject, so they are learning new concepts at the same time they're trying to learn to do proofs.
2. Geometry is fundamentally a right-brained activity, where almost all of the mathematics they've seen before has been left-brained, symbol-manipulation type.
3. At present, it is introduced to everybody, so the average level of mathematical ability in the class is quite low. Thus the proofs tend to be extremely simple, and to repeat the same ideas endlessly.

Anyone who is really going to become a mathematician will have an interest in puzzles, in "how old Mary is when her mother is five times as old as she is now", and in beautiful mathematical patterns and games. They can play with these in math clubs, but they usually won't learn to do rigorous proofs without some help, so perhaps one course on the idea of proof (introduced initially with topics that are familiar like the integers or algebra) as the central concept would be very good.

4 Why we are where we are

People tend to keep doing the same thing. A high school teacher will teach the same things he/she learned. "I learned to do long division, so by God you will too." The "back to basics" folks cause lots of problems in this area. Remember how the ancient Greeks agonized over the invention of writing and how it would destroy the ability of their students to memorize things.

Also, if the curriculum changes, teachers will have to learn the new material, and that may be a lot of work. Thus the teachers themselves, especially the older ones, may be very resistant to change.

Every school district in the country has a set of requirements in their math curricula, and changing them is terribly difficult. Actually, adding things is not difficult, which is why a typical high-school geometry text is 600 pages long and contains almost nothing of interest, while a text from the Soviet Union is 100 pages long and crammed with interesting, meaty problems.

Professional mathematicians write curricula, but in a sense, they are trained primarily to teach other people to become professional mathematicians.

There are huge "turf wars" in universities and (somewhat less) in lower schools. The mathematicians are afraid that if the engineering school teaches math, they'll lose positions. On the other hand, they're not willing to teach that "horrible, ugly, applied math".

Some universities are beginning to see the overwhelming need to teach cross-disciplinary courses. Stanford University, in particular is doing this in a big way, and although it is tough to get started, the results are good. There is no reason this sort of cooperation between departments couldn't be done on a person to person level at any university. A math professor could talk to an engineering professor, and try to coordinate their courses somewhat, for example.

5 Amusing Anecdotes

I can't resist putting in this section. I was able, mostly, to leave personal anecdotes out of the stuff above, but all of these shed some light on various problems in mathematics teaching.

- I was once at a PTA meeting, trying to argue for a bit more emphasis on estimation in the curriculum and was attacked by a furious parent who said that mathematics was not about estimation; it is used only to get exact results, especially concerning that most important topic: money. I asked him how much he would expect to pay monthly if he borrowed $200,000 for a home loan. He said that it's impossible to answer without knowing the interest rate. I said I don't know the rate–the loan is at variable interest–but didn't he think it was worthwhile having some idea of the approximate payments in spite of that? I also asked him this: "If one stick of bubblegum costs a penny, how much would you pay for 1,000,000 sticks? 'A million pennies', is clearly the wrong answer; even the worst businessman in the world can get a better quantity discount than that." Neither argument, of course, had any effect.

- I was an undergraduate at Caltech, and we had a wonderful math curriculum. Of course Caltech had the luxury of having almost all technical people in the classes and all of them took 2 years of math, 2 years of physics, and a year of chemistry. But the courses were synchronized in the sense that when we needed a new mathematical tool in physics, very often that tool had "miraculously" been taught just the week before in the math class. Or the math problem would involve some concept we had just seen in physics. It was great.

- A friend who was educated in the Soviet Union was teaching a university class in linear algebra. Some student asked, "Are there any practical applications of this?" ("this" meaning some topic in linear algebra). My friend thought for a second and said, "Sure, in physics . . . ", but the kid interrupted and said, "I don't know any physics." My friend tried to give examples in chemistry, in engineering, in economics, and in business, but in every case, the kid said, "I don't know anything about (whatever it was)." Finally, my friend said, "You're right. There are no applications. If you don't know anything about the world, mathematics has no applications."

- Here's an anecdote about one of the cross-disciplinary seminars at Stanford that I mentioned earlier. It did not involve pure mathematicians, but rather population biologists and economists. Both groups' eyes were opened. Many of the biolgists' ideas to help prevent ecological destruction were pointed out to be impossible by the economists for reasons that were instantly obvious to the economists, but which had never occurred to the biologists. And the reverse: when one of the biologists said that under certain conditions the entire food production of the world could collapse, one of the economists replied that such a collapse would not be a giant problem. After all, only 3% of the world's economy has to do with food production. . .