# Infinity

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The concept of infinity is fascinating to most people, and as we shall see, it is probably much more interesting than you think.

One of the problems with infinity is that the term has different meanings in different parts of mathematics. It can be used as a limiting value, as in expressions like the following:

$$\zeta(3) = \sum_{i=1}^{\infty} \frac{1}{i^3} \qquad \lim_{x \to 0} \frac{1}{x} = \infty.$$

It can also be used as an artificial "point at infinity" to compactify the real numbers, or there can be an infinite number of points at infinity to build models of projective geometry in various dimensions.

All of the examples above, require a bit of advanced knowledge (except, perhaps, for projective geometry).

Perhaps the way most people first think of infinity is either in terms of listing things: "1, 2, 3, and go on forever", or as a way of counting things: "there are an infinite number of integers". In this paper we'll examine these last two interpretations.

## 1 Counting Finite Sets

Before we plunge into what it means to "count" an infinite number of objects, let's take a quick review of what it means to count a finite number of objects. What does it mean when you say, "This set contains 7 objects"?

The best starting point is to begin by asking what it means for two sets to be the same size. (Mathematicians often say that the sets have the same "cardinality", but what they mean is that the sets are the same size. "Cardinality" is a slightly better word, since it refers only to the count. So a set consisting of three bacteria has the same cardinality as a set consisting of three super-novae, although it's pretty easy to make an argument that the latter set is larger, at least in some sense.)

Here is the formal definition: Two sets, $A$ and $B$, have the same cardinality if there exists a $1-1$ function $f$ that maps set $A$ onto set $B$.

When we say that "$f$ is $1-1$" we mean that the function $f$ never maps two elements of $A$ to the same element of $B$, and since we said it maps "onto" $B$, every element of $B$ is the image of some element of $A$.

This is really just a fancy mathematical way of saying that there is some way to match up the elements in $A$ with those in $B$ so that each object in $A$ corresponds to exactly one object in $B$ and vice versa. Every element has a match in the other set, and nothing is left over.

Generally, there are many, many ways to do this; lots of different functions can be used to demonstrate that the cardinalities of two sets are equal. Luckily, we just need to find 1.

For example, let $A = \{1, 2, 3\}$ and let $B = \{a, b, c\}$. The following function $f$ does the trick (where we are denoting $f$ by a set of ordered pairs as described in the "Something from Nothing" paper available at `http://www.geometer.org/mathcircles/nothing.pdf`). The "$(1, a)$" in the set means that $f$ maps 1 to $a$, or in other words, that $f(1) = a$.):

$$f = \{(1, a), (2, b), (3, c)\}.$$

But there are in reality six different functions that work (in this case):

$$\begin{aligned} f_1 &= \{(1,a),(2,b),(3,c)\} \\ f_2 &= \{(1,a),(3,b),(2,c)\} \\ f_3 &= \{(2,a),(1,b),(3,c)\} \\ f_4 &= \{(2,a),(3,b),(1,c)\} \\ f_5 &= \{(3,a),(1,b),(2,c)\} \\ f_6 &= \{(3,a),(2,b),(1,c)\} \end{aligned}$$

It should be clear that for two finite sets each with $n$ different elements, there are $n!$ different $1-1$ functions that will map one onto the other.

## 1.1 Cardinal Numbers

Of course it's nice to be able to tell if two sets have the same cardinality, but it is nicer still to give that cardinality a name, and so we do. The usual names are "zero", "uno", "due", "tre", "quattro", "cinque", "sei", ... (well, if you're Italian, that is; English speakers use "zero", "one", "two", et cetera). Luckily, almost everybody agrees on the following notation: 0, 1, 2, 3, 4, ..., no matter what language they speak.

In the "Something from Nothing" paper, we defined the set $\mathbb{N}$ of natural numbers from scratch, and completely in terms of set theory: $\mathbb{N} = \{0,1,2,3,4,5,\ldots\}$. But perhaps the nicest thing about the definition was that each of the natural numbers is a set, and not only that; it is a set with the "correct" number of elements (the set corresponding to the number $n$ contains exactly $n$ elements):

$$\begin{aligned} 0 &= \{\} \\ 1 &= \{0\} \\ 2 &= \{0,1\} \\ 3 &= \{0,1,2\} \\ 4 &= \{0,1,2,3\} \\ n &= \{0,1,2,3,\ldots,(n-1)\} \end{aligned}$$

You may also recall the very simple rule we defined to get to the next number:

$$n+1 = n \cup \{n\}.$$

This will be very important later. Notice also that $n \in (n+1)$, and that $n \subset (n+1)$.

Now that we have a collection of cardinal numbers, we can compare our sets with them (well, at least our finite sets). Of course it has to be shown that each natural number has a *different* cardinality and so on, but that's not too difficult. When we find one that matches up with our set, we know how big our set is.

Let's introduce a notation for cardinality. If $S$ is a set, the $\#S$ stands for "the cardinality of $S$", and it will be a cardinal number. For example, $\#\{a,b,c\} = 3$, $\#\{1,3,5,7,\ldots,99\} = 50$, and $\#7 = 7$. (In the last example, remember that 7 is just a set: $7 = \{0,1,2,3,4,5,6\}$.)

We can compare cardinal numbers, of course. $\#S < \#T$ means that the cardinality of set $S$ is strictly smaller than the cardinality of set $T$. Mathematically this means that you can find a function that maps all the elements of $S$ $1-1$ to elements of $T$, but you cannot find such a function that maps to all the values in $T$. In other words, you can find a $1-1$ function that maps $S$ *into* $T$, but you cannot find a $1-1$ function that maps $S$ *onto* $T$. For example, $\#\{1,2,3\} < \#\{1,2,3,4\}$ since no matter how you map the elements of $\{1,2,3\}$ into the set $\{1,2,3,4\}$, something in the latter set will be left out.

## 2 Infinite Cardinal Numbers

Notice that in our definition of when two sets have the same cardinality, we said nothing about whether they were finite or not. In fact the set $\mathbb{N}$ of natural numbers is certainly not finite, but our definition of cardinality allows us to talk about sets that have the same cardinality as $\mathbb{N}$.

We will need a cardinal number to represent the cardinality of $\mathbb{N}$, and what is traditionally used for this is $\aleph_0$, where $\aleph$ (pronounced "aleph") is the first letter in the Hebrew alphabet. The fact that there's a subscript $0$ in $\aleph_0$ is a pretty clear indication that we will come across some other cardinal numbers in the future. In fact, $\aleph_0$ is the smallest infinite cardinal number, $\aleph_1$ is the next larger, $\aleph_2$ the next, and so on[1]. So the notation introduced in the last section can be extended easily: $\#\mathbb{N} = \aleph_0$.

Let's look at a few examples.

The set of all even natural numbers: $\mathbb{E} = \{0, 2, 4, 6, \ldots\}$ is also clearly infinite, but it seems to have only half as many elements as are in $\mathbb{N}$. Surprisingly (perhaps), the sets $\mathbb{N}$ and $\mathbb{E}$ have the same cardinality.

Here's the proof: Let $f(n) = 2n$, and $f$ maps $\mathbb{N}$ into $\mathbb{E}$. It's easy to see that $f$ never maps two numbers to the same number, and that $f$ has all of $\mathbb{E}$ as its range. Thus $\mathbb{N}$ and $\mathbb{E}$ have the same cardinality: $\#\mathbb{N} = \#\mathbb{E} = \aleph_0$.

What if we add an element to $\mathbb{N}$ to make a slightly larger set? Suppose we stick in an element $a$ that's not a natural number to make the set $\mathbb{N}' = \{a, 0, 1, 2, 3, \ldots\}$. What is the cardinality of $\mathbb{N}'$? (To make it definite, let $a = \{\{\{\}\}\}$ which is not a natural number: in fact, $a = \{1\}$.)

The answer is again that it has the same cardinality as $\mathbb{N}$. Here is a function $f$ from $\mathbb{N}$ to $\mathbb{N}'$ that will do the trick:

$$f(x) = \begin{cases} a & : \quad x = 0 \\ x - 1 & : \quad x > 0 \end{cases}$$

Thus $\#\mathbb{N}' = \aleph_0$.

In both cases above, we had sets where one was a strict subset of the other: $\mathbb{E} \subset \mathbb{N} \subset \mathbb{N}'$, but $\mathbb{E} \neq \mathbb{N} \neq \mathbb{N}'$. So one set can be larger than another (in the sense that it contains all the elements of the other and some additional ones), but it can be the same size (from the point of view of cardinality). This is a very important concept; read this paragraph again.

Thus far, all the infinite sets we've looked at are the same size as $\mathbb{N}$. Let's see if we can find one that's larger. A natural place to look might be the rational numbers $\mathbb{Q}$–the set of all fractions.

But again, it turns out that there are the same number of rational numbers as there are natural numbers. We will show that this is the case for the positive rational numbers, and it is not difficult to extend the argument to show that the set of all rational numbers–positive, negative, or zero–can be matched with the natural numbers in a $1-1$ manner.

We begin by making a two-dimensional "list" of all the (positive) rational numbers. In fact, this list has more entries than do the rational numbers because we've listed all possible representations of them–the list contains not only $2/3$, but also $4/6$, $6/9$, $8/12$, and so on:

$$\begin{array}{cccccc} 1/1 & 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 2/1 & 2/2 & 2/3 & 2/4 & 2/5 & \cdots \\ 3/1 & 3/2 & 3/3 & 3/4 & 3/5 & \cdots \\ 4/1 & 4/2 & 4/3 & 4/4 & 4/5 & \cdots \\ 5/1 & 5/2 & 5/3 & 5/4 & 5/5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Now, starting at the upper left corner, number the positions in this two-dimensional table of fractions going back

[1] You may be amazed at what "and so on" actually means!

and forth, advancing along the diagonal as follows:

$$\begin{array}{ccccccc} 0 & 1 & 5 & 6 & 14 & \cdots \\ 2 & 4 & 7 & 13 & 16 & \cdots \\ 3 & 8 & 12 & 17 & 25 & \cdots \\ 9 & 11 & 18 & 24 & 31 & \cdots \\ 10 & 19 & 23 & 32 & 40 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

If you don't understand the numbering scheme, use your pencil to do a "connect the dots": draw a line starting at 0 and then going to 1, 2, 3, 4, 5, 6, and so on. You'll see that the path is boustrophedontic. Also, make sure you realize where the missing numbers in the list are (like 15 or 20, or 42).

So the function $f$ we are looking for that matches all the natural numbers with pairs of natural numbers looks like this:

$$f = \{(0, 1/1), (1, 1/2), (2, 2/1), (3, 3/1), (4, 2/2), (5, 1/3), (6, 1/4), (7, 2/3), \ldots\}.$$

As we warned above, this isn't *exactly* what is needed: a true mapping would match one integer to every rational number that had not yet been listed. The following enumeration does exactly that– an "$X$" in the table indicates that nothing is mapped to that particular fraction:

$$\begin{array}{ccccccc} 0 & 1 & 4 & 5 & 10 & \cdots \\ 2 & X & 6 & X & 12 & \cdots \\ 3 & 7 & X & 13 & 19 & \cdots \\ 8 & X & 14 & X & 23 & \cdots \\ 9 & 15 & 18 & 24 & X & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

With a numbering scheme like the one above, it is clear that it is possible to construct a $1-1$ mapping from the natural numbers onto the positive rational numbers. It's a nice exercise to convert this function to one that maps the natural numbers $1-1$ onto the entire set $\mathbb{Q}$ of rational numbers. Therefore $\#\mathbb{Q} = \aleph_0$.

But we're still not making any progress–we tried to find a larger set, and the rationals seemed like they might be–but in fact the cardinality of the rationals is the same as the cardinality of the natural numbers.

Let's try again. Let's let $\mathbb{P}$ be the set of all polynomials with natural number coefficients[2].

$\mathbb{P}$ thus consists of all the polynomials that look like these:

$$\begin{array}{l} 1 \\ 3x + 17 \\ x^2 + 3x + 41 \\ x^5 + 5x^4 + 3x^3 + 7x^2 + 121x + 11213 \\ x^{100020102} + 4356654x^{100020101} + \cdots + 17x + 41 \end{array}$$

We'll allow polynomials with any number of terms, and we'll lump them all into one giant set. Surely this set has more elements than the integers, right?

Nope: there are the same number of items in both sets.

What follows is an amazingly simple way to map all the polynomials above into the integers so that nothing is left out. It is all based on the idea of prime factorization of the integers. (Our proof will work for polynomials

[2]Note that it wouldn't help to make it be the set of all polynomials with rational coefficients since we already know that there are the same number of rational numbers as there are of natural numbers.

with non-negative coefficients. Again, it is no big deal to extend it to the general case, but things are slightly uglier.)

Recall that any natural number can be written in a unique way as the product of prime numbers. For convenience, let's assign a nice series of names to the primes: $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, $p_4 = 11$, $p_5 = 13$, and so on. Thus, $p_i$ is the $i^{\text{th}}$ prime number, where we begin numbering from zero.

Now, if $k_0, k_1, k_2, \ldots$ is a series of integers where all but a finite number of them are zero, then every different sequence of the $k_i$ represents a different integer of the form:

$$p_0^{k_0} p_1^{k_1} p_2^{k_2} p_3^{k_3} p_4^{k_4} \cdots = \prod_{i=0}^{\infty} p_i^{k_i}.$$

Notice that we can run the infinite product above to infinity since almost all the $k_i$ are equal to zero, so that almost all of the $p_i^{k_i}$ will be 1. And the really great thing is that for any different sequence of $k_i$, the product above will generate a different integer, and all the integers are generated by some series of $k_i$.

Let's write down the most general polynomial:

$$k_0 + k_1 x + k_2 x^2 + k_3 x^3 + k_4 x^4 + \cdots = \sum_{i=0}^{\infty} k_i x^i,$$

where all but a finite number of the $k_i$ are equal to zero.

But doesn't this provide an exact, $1-1$ mapping between the polynomials with coefficients in the natural numbers and the natural numbers themselves? Yes–so there are the same number of polynomials (of any degree) as there are natural numbers: $\#\mathbb{P} = \aleph_0$. We still have not succeeded in constructing a set or cardinality larger than $\mathbb{N}$.

## 2.1 A Larger Cardinal Number

Remember the axiom of the power set from the previous lecture? It states that if you start with any set, there exists a set consisting of all of the subsets of that set, called the power set of the original set. For example, if the set you begin with is $S = \{0, 1, 2\}$, then the following set, consisting of all its subsets, is also a set:

$$\mathcal{P}(S) = \{\{\}, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

We use the notation $\mathcal{P}(S)$ to indicate the "power set of $S$".

In general, if $S$ is a finite set of cardinality $n$, then the cardinality of $\mathcal{P}(S)$ is $2^n$. That's because for every element $x$ of $S$, and every subset $T \subset S$, either $x \in T$ or $\notin T$, and since that decision has to be made for every element $x$ of $S$, the size of the power set is $2^n$. Check the example above to see that the cardinality of $(\{0, 1, 2\}) = 2^3 = 8$.

But there was nothing in the power set axiom that required the set to be finite. We can, for example, look at $\mathcal{P}(\mathbb{N})$. How big is it? We will show that $\#\mathcal{P}(\mathbb{N}) > \#\mathbb{N}$.

How can we possibly do this? How do we show that no matter what function you examine, it will not match up the elements of $\mathbb{N}$ with the elements of $\mathcal{P}(\mathbb{N})$?

Here's how it is done. Suppose that there is some way to match up the elements. We will show that whatever scheme you use, something is left out.

The nice thing about subsets is that to identify the subset, all you have to know for each element in the original set is whether it is in the subset or not. Thus, we can write down a complete description of a subset of $\mathbb{N}$ in the following form:

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| S | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | $\cdots$ |

Wherever there is a 1, the natural number is in the set; wherever there is a 0, it is not in the subset. So in the example above, the set $S$ contains 1, 2, 6, 9, 10, 11, and perhaps some other entries beyond number 12.

As a concrete example, the complete list of subsets of the finite set $\{0, 1, 2\}$ that we considered above in this form:

| | 0 | 1 | 2 |
|---|---|---|---|
| $\{\}$ | 0 | 0 | 0 |
| $\{0\}$ | 1 | 0 | 0 |
| $\{1\}$ | 0 | 1 | 0 |
| $\{2\}$ | 0 | 0 | 1 |
| $\{0, 1\}$ | 1 | 1 | 0 |
| $\{0, 2\}$ | 1 | 0 | 1 |
| $\{1, 2\}$ | 0 | 1 | 1 |
| $\{0, 1, 2\}$ | 1 | 1 | 1 |

If we want to list a series of subsets, we would form a table something like the following, where $S_0$, $S_1$, $S_2$, et cetera, are the sets. (For now, ignore the boxes surrounding some of the numbers.) In this example, the subset $S_0$ contains 1, 2, 6, 9, 10, 11, ...; $S_1$ contains 0, 1, 4, 6, 9, 11, 12, ..., and so on.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_0$ | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | $\cdots$ |
| $S_1$ | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | $\cdots$ |
| $S_2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | $\cdots$ |
| $S_3$ | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | $\cdots$ |
| $S_4$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | $\cdots$ |
| $S_5$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

Suppose there is some way to match up the natural numbers with all the subsets of the natural numbers. Then there will be some function that maps every natural number $n$ into some subset $S_n$ of the natural numbers in such a way that every subset is listed exactly once.

If this were possible, we could make a listing of the subsets as in the table above where all the slots are filled with 0 or 1. We will now show that no matter how the list is arranged, at least one of the subsets has been left out. To illustrate, suppose the table above gives a function that works (matches subsets of $\mathcal{N}$ $\mathcal{N}$). Consider all the elements on the diagonal (they have boxes around them in the figure), and make a new subset $S_m$ that has a 1 wherever the diagonal is 0, and a 0 wherever it is one. For the example above, $S_m$ would look like this:

| | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $S_m$ | 1 | 0 | 0 | 0 | 1 | 1 | $\cdots$ |

This $S_m$ is missing (the "$m$" subscript stands for "missing") from the list. It couldn't possibly be $S_0$ because it differs from $S_0$ for element 0. Similarly, it couldn't be $S_1$ since it differs from $S_1$ for element 1, and so on. Thus, no matter how you number the subsets, at least one of them will be left out, so we know that $\#\mathcal{P}(\mathbb{N}) > \#\mathbb{N} = \aleph_0$.

This method above can be used in other settings, and is usually called "Cantor's diagonalization method", since you go down the diagonal and change whatever you find there. Georg Cantor was the mathematician who did much of the fundamental work with cardinal and ordinal numbers (and he happened to speak Hebrew, which accounts for the Hebrew letter $\aleph$ as part of the name of infinite cardinal numbers).

How much bigger is $\#\mathcal{P}(\mathbb{N})$ than $\aleph_0$? Is $\#\mathcal{P}(\mathbb{N}) = \aleph_1$, the next larger cardinal? Or are there cardinal numbers between $\aleph_0$ and $\#\mathcal{P}(\mathbb{N})$? The long answer is far too difficult to discuss here[3], but the short answer (without proof) is that the set-theoretic statement that $\#\mathcal{P}(\mathbb{N}) = \aleph_1$ is undecidable using the axioms of Zarmelo-Frankel set theory. There is no way to prove that it is true, and no way to prove that it is false. You can add it as an axiom to the set of Z-F axioms, and if the axioms are consistent by themselves, they will continue to be consistent with the addition of this as an axiom. But you can also add the negation of this statement as an axiom, and that is also perfectly consistent.

The statement that $\#\mathcal{P}(\mathbb{N}) = \aleph_1$ is called the "continuum hypothesis" because the cardinality of the collection of subsets of $\mathbb{N}$ turns out to be the cardinality of the real numbers, $\mathbb{R}$, and one often refers to the continuum of real numbers, since they are packed together continuously along the real line.

In fact, if you read most explanations of set theory, you'll find that the first example given of a set with cardinality larger than the natural numbers is the set of real numbers. The proof is quite similar to what we did above. Here's roughly how it goes.

First, we'll just show that the cardinality of a small part of the reals is still larger than the cardinality of the natural numbers; we will just look at the real numbers $x$ such that $0 \leq x < 1$. Every one of those numbers has an infinite decimal expansion, for example $1/2 = .500000\ldots$, $1/3 = .33333\ldots$, $1/7 = .142857142857\ldots$, $\pi - 3 = .1415926535\ldots$, $\sqrt{2} - 1 = .4142135\ldots$, and so on.

As before, we assume there is some way to number them all with the natural numbers, and we could form a list:

$$
\begin{aligned}
0 &\rightarrow .50000000000000000000000000000000000000000000000000000000000\ldots \\
1 &\rightarrow .33333333333333333333333333333333333333333333333333333333333\ldots \\
2 &\rightarrow .14285714285714285714285714285714285714285714285714285714285\ldots \\
3 &\rightarrow .14159265358979323846264338327950288419716939937510582097494\ldots \\
4 &\rightarrow .41421356237309504880168872420969807856967187537694807317668\ldots
\end{aligned}
$$

and so on.

Then, to show that at least one real number was left out, go down the diagonal and make up a new number that differs in the $i^{\text{th}}$ decimal place from the real number corresponding to $i$ (where we number the decimal places from number zero). So if the above list were supposed to be complete, we merely need to find a number that has something other than a 5 in the zeroth slot, other than a 3 in the first slot, other than a 2 in the second slot, and so on. In fact, we could just add 1 to each unless it was a 9, in which case it would wrap back to zero. The first five decimal places of a real number guaranteed to be missing from the list above with that strategy would be $.64362\ldots$.

In fact, if we had written the numbers not as decimal expansions, but as binary expansions, our proof would be virtually identical with the proof using subsets, since the binary expansions would just be series of zeroes and ones.

A minor annoyance that complicates both of the above examples (although it is not hard to fix) is that many numbers (in fact, infinitely many) have two different decimal (or binary) expansions. For example, $.3999999\ldots = .4$, $.1314999999\ldots = .1315$, et cetera (where we assume, of course that the 9's continue forever). For one thing, we must be certain that only one version is listed for every real number on the right, and for another, that the resulting infinite decimal is not one of the ambiguous ones, and hence appearing elsewhere on the list.

In any case, it is true that $\#\mathbb{R} = \#\mathcal{P}(\mathbb{N})$.

[3]Paul Cohen proved, using a method called "forcing", that this so-called "continuum hypothesis" is independent of the other axioms of Z-F set theory. His proof fills an entire book.

# 3 Still Larger Cardinals

Cantor's diagonalization process can be used with any set $S$, finite or infinite, to show that $\#S < \#\mathcal{P}(S)$. In other words, the collection of subsets of a given set always has a larger cardinality than the cardinality of the original set.

We have already seen that if $S$ is finite, then if $\#S = n$, then $\#\mathcal{P}(S) = 2^n$. We will simply use the same notation for infinite cardinals, and we have shown above that $\#\mathbb{R} = 2^{\aleph_0}$.

But the cardinality of the collection of all subsets of $\mathbb{R}$ is larger than the cardinality of $\mathbb{R}$ itself, so we've got an even larger cardinal, and there is no reason we cannot continue this for as long as we want. For any cardinal that exists, you can construct a larger one simply by taking its power set.

Thus we have an infinite collection of cardinal numbers:

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, \ldots.$$

So that must be all of them, right?

No!

Using the axioms of set theory, it is easy to show that the following is also a valid set:

$$\{\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, \ldots\}.$$

And another axiom of set theory tells us that we can take the union of all the elements of a valid set and obtain another valid set, so

$$\aleph_0 \cup \aleph_1 \cup \aleph_2 \cup \ldots$$

is also a valid set[4].

How big is it? Well, for every $i$, it must be larger than $\aleph_i$, since we know that there are more than $\aleph_{i+1}$ elements in it.

This is, in fact, the smallest "limit cardinal". We can write it as:

$$\aleph_{\aleph_0}.$$

But you can see where this is going to lead–the power set of $\aleph_{\aleph_0}$ is larger still, and we can keep applying the power set operations to the resulting sets, and then take the union of all of those, for a second limit cardinal. And a third, and a forth, et cetera, an infinite number of times. Well, why not take the union of all of those. Eventually, we'll get to:

$$\aleph_{\aleph_1}.$$

But why stop here? Continue to $\aleph_{\aleph_2}$, $\aleph_{\aleph_3}$, and so on. Then take *their* union. What is this:

$$\aleph_{\aleph_0} \cup \aleph_{\aleph_1} \cup \aleph_{\aleph_2} \cup \aleph_{\aleph_3} \cup \ldots?$$

It is obviously:

$$\aleph_{\aleph_{\aleph_0}}.$$

There is literally no end to the process. We can form:

$$\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \aleph_{\aleph_{\aleph_{\aleph_0}}}, \aleph_{\aleph_{\aleph_{\aleph_{\aleph_0}}}}, \ldots$$

and then we can union all of them together, et cetera.

[4]The sharp-eyed reader will notice that we have cheated a little bit here, and assumed that for each cardinal number we actually have a set of that size, in the same way that we had sets for all the finite cardinals, with the set "7" being a set of 7 elements. It is possible to do this, but in this paper we have not shown how. The section on ordinal numbers (see section 4) will give some indication of how this is done.

### 3.1 The Schröder-Bernstein Theorem

Although we will not prove it here[5], the Schröder-Bernstein Theorem shows, in a sense, that all the analysis above makes sense.

We have already seen that it is difficult sometimes to construct a function that maps a set perfectly $1-1$ onto another one. For example, our first numbering of the rationals had every possible version of every fraction: $1/2$, $2/4$, $3/6$, et cetera, in the table, and we had to go to some trouble to get rid of the duplicates to show that there was really an exact matching function.

The Schröder-Bernstein Theorem allows us to be lazy. It says that if $S$ and $T$ are any two sets (of any cardinality), and if you can find a function $f$ that maps $S$ $1-1$ *into* (not necessarily *onto*) $T$, and you can find a functions $g$ that maps $T$ $1-1$ *into* (again, not necessarily *onto*) $S$, then $\#S = \#T$. In other words, the existence of such functions $f$ and $g$ guarantee the existence of another function $h$ that maps $S$ $1-1$ and *onto* $T$.

## 4 Ordinal Numbers

If you paid attention in your English language classes, you learned the difference between the cardinal numbers and the ordinal numbers. The cardinal numbers are "one", "two", "three", and so on, and the ordinal numbers are "first", "second", "third", and so on.

Mathematicians don't bother with this distinction, and since we usually write the numbers as "23" instead of "twenty three", for a mathematician, the cardinal numbers are 0, 1, 2, 3, ..., and the ordinal numbers are 0, 1, 2, 3, .... And there's another difference: English teachers tend to start at 1 and mathematicians start at zero.

So they're the same, right? Well, no. All the *finite* ones are the same, but then there's a difference. In this section, we'll look at the construction of the infinite ordinals.

The way that both English and mathematics use the ordinal numbers and the cardinal numbers is similar, however. The cardinal numbers are used to count things, and the ordinal numbers are used to order things. Thus the "one", "two", "three" versus "first", "second", "third" distinction in English makes good sense.

What we intend to do is follow the same general plan that we have used up to now, with a few hints taken from how we treated the cardinal numbers. Once we had a definite ordinal number $n$, we could make the next one, which we called "$n+1$" by letting $n+1 = n \cup \{n\}$. This works fine, and in the case of the finite numbers, it can go on forever, but it will never get us to infinity: we can keep making numbers that are larger and larger, but every particular number we make is still finite. To get to the first infinite cardinal number, $\aleph_0$, we needed to somehow take the union of all the numbers up to that point:

$$\aleph_0 = 0 \cup 1 \cup 2 \cup 3 \cup 4 \cup \cdots$$

Notice how this makes perfect sense:

$$\aleph_0 = \{\} \cup \{0\} \cup \{0,1\} \cup \{0,1,2\} \cup \{0,1,2,3\} \cup \cdots$$

In other words,

$$\aleph_0 = \{0,1,2,3,4,\ldots\} = \mathbb{N},$$

so the first infinite cardinal, $\aleph_0$, is the same as $\mathbb{N}$.

Since $\aleph_0 = \mathbb{N}$ is the first infinite cardinal, we will see that it also makes sense to make it the first infinite ordinal, and (again for reasons that will become clearer later on), we will give it a third name if we want to think of it as an ordinal: we'll call it $\omega$–the Greek letter "omega"–so $\aleph_0 = \mathbb{N} = \omega$. Omega is the last letter of the Greek alphabet, so "from alpha to omega" means "from the beginning to the end: omega is the end. As we shall see later, this is quite a joke.

[5]For a proof, see Patrick Suppe's Book entitled, "*Axiomatic Set Theory*".

## 4.1 Constructing the Infinite Ordinals

If $\omega$ is the first infinite ordinal, why not go ahead and make the next ordinal as usual: $\omega + 1 = \omega \cup \{\omega\}$? In fact, that's just what is done.

As soon as we have $\omega + 1$, we can construct $\omega + 2$: $\omega + 2 = (\omega + 1) \cup \{\omega + 1\}$. Similarly, we can construct $\omega + 3$, $\omega + 4$, et cetera.

How big are $\omega + 1$, $\omega + 2$, et cetera? In other words, what is the cardinality of these sets? Actually, we already worked out the cardinality of $\omega + 1$ in section 2–we found the cardinality of $\mathbb{N} \cup \{a\}$, where $a$ is any object not in $\mathbb{N}$–it is just $\aleph_0$.

Similarly, it is not hard to show that $\#(\omega + 1) = \#(\omega + 2) = \#(\omega + 3) = \cdots = \aleph_0$.

Since we have constructed, as sets, $\omega + 1$, $\omega + 2$, $\omega + 3$, and so on, we can union them all together to get:

$$\omega + 1 \cup \omega + 2 \cup \omega + 3 \cup \cdots = \omega \times 2,$$

where "$\times$" can be interpreted as multiplication.

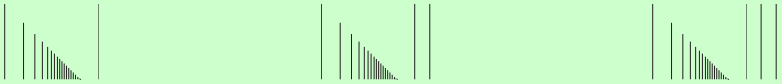
From there, it is a simple matter to continue to $\omega \times 2 + 1$, $\omega \times 2 + 2$, et cetera. These can all be unioned together to make $\omega \times 3$. Similarly, we can make $\omega \times 4$, $\omega \times 5$, et cetera.

If we union all of $\omega$, $\omega \times 2$, $\omega \times 3$, et cetera, we can get to $\omega \times \omega = \omega^2$.

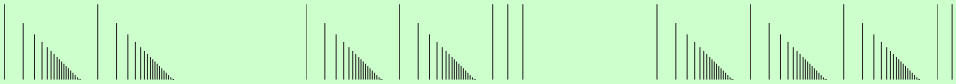
There is a nice way to visualize these ordinals[6].). Begin by imagining that each ordinal is represented by a row of telephone poles. So the ordinal 5 is represented by 5 telephone poles in a row, et cetera. This works fine for any finite number, but if you would like to visualize $\omega$, it's an infinite line of poles, beginning at zero, and going on forever. Since it's difficult to draw an infinite number of them on a finite page, we'll imagine them "drawn in perspective", disappearing toward the horizon. So 3 telephone poles (corresponding to the ordinal number 3) looks like this:

Then $\omega$ of them would look like this:

Given this visualization method, here are what $\omega + 1$, $\omega + 2$, and $\omega + 3$ look like:

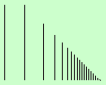


Continuing with a couple of other examples, here is what $\omega \times 2$, $\omega \times 2 + 3$, and $\omega \times 3 + 2$ look like:



To add two ordinals together, you just draw the pictures next to each other and see what the resulting picture looks like. We must be a bit careful (and this is where the telephone pole example helps) in the order that we add them. If you already have an infinite number of poles going off to the horizon (numbered from zero on up), and you add a pole in front of pole number zero (sort of like pole number $-1$), then it's really not going to look any different from what you had before: it's just an infinite number of telephone poles starting from a fixed pole. Thus, we have $1 + \omega = \omega$. Here's the picture you get by drawing a 1 followed by an $\omega$:

[6]This is due to John Conway: see his book written with Richard Guy entitled "*The Book of Numbers*"



It is just like the drawing for $\omega$ except that we've drawn the perspective a little wrong.

$\omega + 1$, however, is clearly different: there's a new pole placed "after" the horizon. The new pole you placed has nothing after it, while every pole in the original $\omega$ had something after it. It's easy to see that adding 2 or 3 or even 1000000 poles in front of $\omega$ make no change in the general drawing:

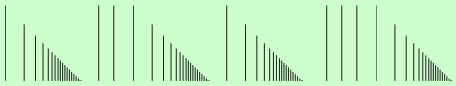
$$2 + \omega = 3 + \omega = 1000000 + \omega = \omega,$$

but

$$\omega + 2 \neq \omega + 3 \neq \omega + 1000000 \neq \omega.$$

Thus ordinal arithmetic is not commutative: if $\alpha$ and $\beta$ are two ordinals, it is not necessarily the case that $\alpha + \beta = \beta + \alpha$. Sometimes it is: $2 + 5 = 5 + 2$, or $\omega + \omega \times 2 = \omega \times 2 + \omega$, and sometimes it is not: $\omega + 1 \neq 1 + \omega$.

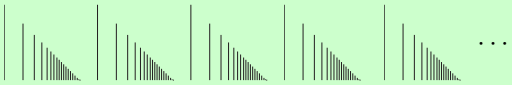
So what is $(\omega + 2) + (\omega \times 2 + 3) + \omega$? Draw them out and see:



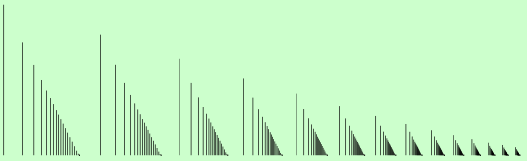
which is clearly equal to $\omega \times 4$, with a few extra poles that get sucked up by intermediates $\omega$s.

Of course we've been a little sloppy; we've used the notation $\omega \times 2$, but what does it mean? $\omega \times 2$ is 2 copies of $\omega$, and $2 \times \omega$ is $\omega$ copies of 2. We know what two copies of $\omega$ look like: just two sets of poles disappearing into the horizon, but $\omega$ copies of 2 just amounts to putting down 2 poles, then 2 more, then 2 more, et cetera, $\omega$ times. So $2 \times \omega = \omega \neq \omega \times 2$

If we take the union of $\omega$, $\omega \times 2$, $\omega \times 3$, $\omega \times 4$, we clearly get $\omega$ copies of $\omega$, which we would write as $\omega \times \omega$, or as $\omega^2$, looking something like this:



It would perhaps be even better if we drew each of the $\omega$s so that they tended to disappear into a second horizon, et cetera:



But then, as before, we can look at $\omega^2 + 1$, $\omega^2 + 2$, …, $\omega^2 + \omega$, $\omega^2 + \omega + 1$, …, $\omega^2 + \omega \times 2$, …, $\omega^2 + \omega \times 3$, …, $\omega^2 + \omega^2 = \omega^2 \times 2$, …, $\omega^2 \times 3$, …, $\omega^2 \times 4$, …, $\omega^3$, …, $\omega^4$, …, $\omega^5$, …, $\omega^\omega$, $\omega^\omega + 1$, ….

Although we haven't proved it, all of the ordinals listed in the previous paragraph have the same cardinality: $\aleph_0$. This may seem wrong, but it's true. In fact, when we were looking at our first examples of cardinal numbers, we essentially proved a bunch of the ordinals above to have the same cardinality.

Look at $\omega + 1$. Isn't that like the set $\{a, 0, 1, 2, 3, \ldots\}$ which we called $\mathbb{N}'$ in section 2? How about the rational numbers $\mathbb{Q}$? A little thought should convince you that $\mathbb{Q}$ looks a heck of a lot like $\omega^2$. And how about $\mathbb{P}$, the set of all polynomials with integer coefficients? Again, a little thought will show you that $\mathbb{P}$ looks just like

$$\omega + \omega^2 + \omega^3 + \cdots = \omega^\omega.$$

The first $\omega$ is the constant polynomials; the $\omega^2$ is the size of the polynomials of the form $ax+b$; the $\omega^3$ corresponds to the set of polynomials of the form $ax^2 + bx + c$, et cetera.

It begins to boggle the mind, doesn't it? But let's do one last boggle before ending this paper: it's essentially equivalent to what mathematicians call the "well-ordering principle". Imagine *any* ordinal, as complex as you want. Start from there, and count down. By counting down is meant simply writing a sequence of decreasing ordinals: you don't have to (and you can't) hit all of them. What is truly mind-boggling is that no matter how you choose to count down, your entire list has to be finite. Play with it and see!